Krull-like domains arising from the Anderson rings

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Questions

Can we find a condition under which
the Anderson rings become Krull-like domains?

Table of Contents

Construction of the Anderson rings

Localization at α

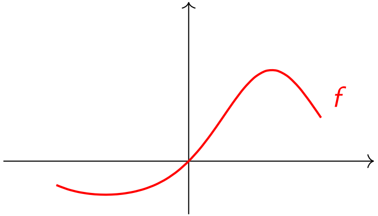
Let K be a field and let $K[X]$ be the polynomial ring over K.

- Consider the point $\alpha \in K$.
- $M_\alpha = (X - \alpha)$ is a maximal ideal of $K[X]$.
- We obtain the ring $K[X]_{M_\alpha}$,
  which is called the *localization* at α in $K[X]$.
- In this case, $K[X] \setminus M_\alpha = \{f \in K[X] \mid f(\alpha) \neq 0\}$.

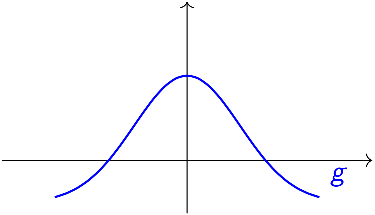
Localization at 0

Consider the localization at 0 in $K[X]$.

- $K[X] \setminus M_0 = \{f \in K[X] \mid f(0) \neq 0\}$
  $= \{f \in K[X] \mid f(0) \text{ is a unit in } K\}$.



$\frac{1}{f} \notin K[X]_{M_0}$



$\frac{1}{g} \in K[X]_{M_0}$

Generalization of ‘localization at 0’: Anderson rings

Consider the set $A = \{f \in R[X] \mid f(0) = 1\}$.

- A is a regular multiplicative subset of $R[X]$, whose saturation is $\{f \in R[X] \mid f(0) \text{ is a unit in } R\}$.
- We obtain the quotient ring $R[X]_A$.

In this case, $R[X]_A$ is called the *Anderson ring* of R.

Daniel D. Anderson (1948 - 2022)

Nagata rings and Serre's conjecture rings

Let R be a commutative ring with identity.
Consider the regular multiplicative sets of $R[X]$

$$N = \{f \in R[X] \mid c(f) = R\} \text{ and } U = \{f \in R[X] \mid f \text{ is monic}\}$$
$$\widetilde{U} = \{f \in R[X] \mid \text{the coefficient of lowest term in } f \text{ is } 1\}.$$

- We call $R[X]_N$ the *Nagata ring* of R and
  $R[X]_U$ the *Serre's conjecture ring* of R.
- $R[X]_A \subseteq (R[X]_A)[\frac{1}{X}] = R[X]_{\widetilde{U}} \cong R[X]_U \subseteq R[X]_N$

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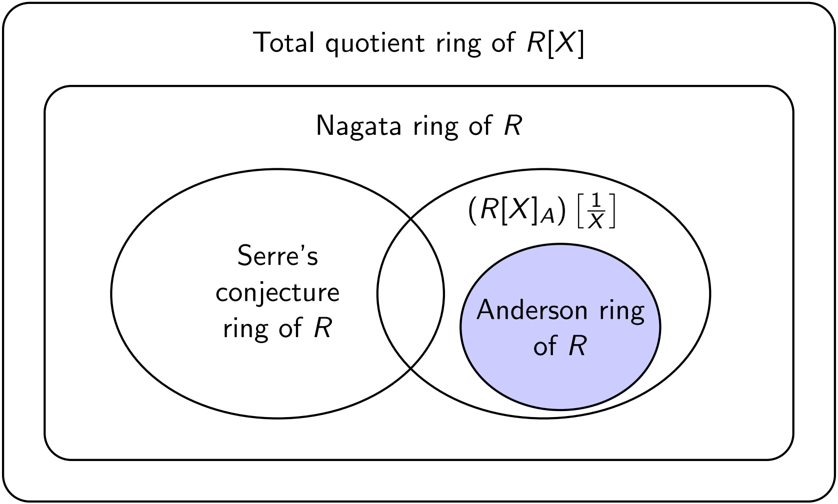
Total quotient ring of $R[X]$
Nagata ring of R
Serre's conjecture ring of R
$(R[X]_A)\left[\frac{1}{X}\right]$
Anderson ring of R

Some generalizations of Krull-domains

Krull domains

Let D be an integral domain and
let $X^1(D)$ be the set of height one prime ideals of D.

- D is a *Krull domain* if D satisfies following three condition:
  (1) $D = \bigcap_{P \in X^1(D)} D_P$
  (2) D_P is a quasi-local PID which is not a field for all $P \in X^1(D)$
  (3) every nonzero element of D belongs to only finite number of $P \in X^1(D)$.

Krull-like domains

Let D be an integral domain and
let $X^1(D)$ be the set of the height-one prime ideals of D.
Consider the following conditions:

(i) $D = \bigcap_{P \in X^1(D)} D_P$.

(ii) D_P is a valuation domain for each $P \in X^1(D)$.

(iii) D_P is a Noetherian domain for each $P \in X^1(D)$.

(iv) Every nonzero element x of D belongs to only finitely many $P \in X^1(D)$.

Based on these statements, we define the following:

1. D is a *weakly-Krull domain* if it satisfies (i) and (iv).
2. D is an *infra-Krull domain* if it satisfies (i), (iii) and (iv).
3. D is a *generalized Krull domain* (in the sense of Gilmer) if it satisfies (i), (ii) and (iv).

Maximal t-ideals

Let D be an integral domain with quotient field K and let I be a nonzero ideal of D.

- $I^{-1} = \{a \in K \mid aI \subseteq D\}$
- $I_v = (I^{-1})^{-1}$
- $I_t := \bigcup\{J_v \mid J \text{ is a nonzero finitely generated subideal of } I\}$
- I is a *t-ideal* of D if $I_t = I$
- I is a *maximal t-ideal* if there does not exist a proper t-ideal which properly containing I, and denoted by $I \in t\text{-}\mathrm{Max}(D)$.

Krull-like domains II (GK-domains)

Let D be an integral domain.

- A valuation domain D is a *strongly discrete valuation domain* if every nonzero prime ideal of D is not idempotent.
- D is a *GK-domain* if
  D_M is a strongly discrete valuation domain for any $M \in t\text{-}\mathrm{Max}(D)$ and every principal ideal has only finitely many minimal prime ideals.

Well-known facts of Krull-like domains

- Weakly Krull domains are exactly
  a one t-dimensional domains with finite t-character.
- Infra-Krull domains are exactly
  a one t-dimensional strong Mori domain.
- Generalized Krull domains are precisely
  a one t-dimensional PvMD with finite t-character.
- GK-domains are exactly a PvMD that is also a t-SFT-ring.

Krull-like domains arising from Anderson rings

Can we find a condition under which the Anderson rings become Krull-like domains?

Krull domains arising from Anderson rings

Let D be an integral domain.

- If D is a Krull domain,then so is D_S for multiplicative subset S of D.
- If D is a Krull domain, then so is $D[X]$.
- If $D[X]_N$ is a Krull domain, then so is D.

Remark (Trivial result)

Let D be an integral domain.
Then D is a Krull domain if and only if $D[X]_A$ is a Krull domain.

- Finite t-character
- Strong Mori domain
- One t-dimensional integral domain
- PvMD
- t-SFT-ring

Maximal t-ideals of Anderson rings

Theorem (Submitted, Baek and Lim)

Let D be an integral domain.
If $\mathfrak{m}$ is a maximal t-ideal of $D[X]_A$, then $\mathfrak{m}$ is exactly of the form

(1) $MD[X]_A$ for some maximal t-ideal M of D, or
(2) $\mathfrak{p}D[X]_A$, where $\mathfrak{p} \in t\text{-}\mathrm{Max}(D[X])$ is an upper to zero in $D[X]$ disjoint from A.

In addition, if D is integrally closed,
then the type (1) and (2) are the only maximal t-ideals of $D[X]_A$.

finite t-character

Let D be an integral domain.

- D has *finite t-character* if
  any nonzero nonunit element of D is contained in
  only a finite number of maximal t-ideals of D.

Proposition (Submitted, Baek and Lim)

Let D be an integral domain.
Then D has finite t-character if and only if $D[X]_A$ has finite t-character.

- Finite t-character
- Strong Mori domain
- One t-dimensional integral domain
- PvMD
- t-SFT-ring

t-local properties of Anderson ring

Let D be an integral domain and let (P) be a property of integral domains.

Suppose that property (P) satisfies following two conditions:

- every DVR has a property (P)
- D has a property (P) if and only if $D[X]_N$ has a property (P).

Then we obtain

- D_M has a property (P) for all $M \in t\text{-}\mathrm{Max}(D)$ if and only if $(D[X]_A)_{\mathfrak{m}}$ has a property (P) for all $\mathfrak{m} \in t\text{-}\mathrm{Max}(D[X]_A)$.

Strong Mori domains

Let D be an integral domain.

- D is a *strong Mori domain* if
  D is a t-locally Noetherian domain which has finite t-character.

Proposition (Submitted, Baek and Lim)

Let D be an integral domain.
Then D is a strong Mori domain if and only if
$D[X]_A$ is a strong Mori domain.

- Finite t-character
- Strong Mori domain
- One t-dimensional integral domain
- PvMD
- t-SFT-ring

t-dimensions

Let D be an integral domain and
let $\mathcal{P} := \{P_\alpha \mid \alpha \in \Lambda\}$ be a chain of prime t-ideals of D.

- The *t-(Krull-)dimension* of D is $\max\{\mathrm{leng}(\mathcal{P}) \mid \mathcal{P}$ is a chain of prime t-ideals of $D\}$, and denoted by t-dim(D).

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The t-dimension of Anderson rings over UMT-domains

Let D be an integral domain.

- D is a *UMT-domain* if
  every upper to zero in $D[X]$ is a maximal t-ideal of $D[X]$.

Proposition (Submitted, Baek, Lim and Tamoussit)

Let D be an integral domain that is not a field.
If $t\text{-dim}(D[X]_A) = 1$, then $t\text{-dim}(D) = 1$.
Moreover, the converse holds when D is a UMT-domain.

- Finite t-character
- Strong Mori domain
- One t-dimensional integral domain
- PvMD
- t-SFT-ring

Prüfer v-multiplication domains

Let D be an integral domain.

- An ideal I of D is *t-invertible* if $(II^{-1})_t = D$.
- D is a *Prüfer v-multiplication domain* (for short, PvMD) if every finitely generated ideal is t-invertible.

PvMD arising from Anderson rings

Let D be an integral domain.

- If D is a PvMD,then so is D_S for any multiplicative subset S of D.
- If D is a PvMD, then so is $D[X]$.
- If $D[X]_N$ is a PvMD, then so is D.

Remark (Trivial result)

Let D be an integral domain.
Then D is a PvMD if and only if $D[X]_A$ is a PvMD.

- Finite t-character
- Strong Mori domain
- One t-dimensional integral domain
- PvMD
- t-SFT-ring

t-SFT-rings

Let D be an integral domain and let I be an ideal of D.

- I is a *t-SFT-ideal* if
  there exists a finitely generated ideal J contained in I and a positive integer $n \in \mathbb{N}$ such that $a^n \in J_v$ for all $a \in I_t$.
- D is a *t-SFT-ring* if
  every nonzero ideal of D is a t-SFT-ideal.

Proposition (Submitted, Baek, Lim and Tamoussit)

Let D be an integral domain.
If $D[X]_A$ is a t-SFT-ring, then so is D.
In addition, the converse holds when D is integrally closed.

- Finite t-character
- Strong Mori domain
- One t-dimensional integral domain
- PvMD
- t-SFT-ring

Well-known facts of Krull-like domains

- Weakly Krull domains are exactly
  a one t-dimensional domains with finite t-character.
- Infra-Krull domains are exactly
  a one t-dimensional strong Mori domain.
- Generalized Krull domains are precisely
  a one t-dimensional PvMD with finite t-character.
- GK-domains are exactly a PvMD that is also a t-SFT-ring.

Corollary (Submitted, Baek, Lim and Tamoussit)

Let D be an integral domain. Then the following assertions hold.

(1) If $D[X]_A$ is a weakly Krull domain, then so is D.
Moreover, the converse holds when D is a UMT domain.

(2) $D[X]_A$ is an infra-Krull domain if and only if D is an infra-Krull domain.

(3) $D[X]_A$ is a generalized Krull domain if and only if D is a generalized Krull domain.

(4) $D[X]_A$ is a GK domain if and only if D is a GK domain.

Related papers

- A special subring of the Nagata ring and Serre's conjecture ring
  - H. Baek and J. W. Lim
  - Submitted
  - 10.48550/arXiv.2408.08758
- On the transfer of certain ring-theoretic properties in Anderson rings
  - H. Beak, J. W. Lim and A. Tamoussit
  - Submitted
  - 10.48550/arXiv.2410.17007

Thank you for your attention!!

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