A generalization of an irreducibility theorem of Schur

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Schur's Theorem (1930)

Theorem

For every $n \geq 1$, the truncated exponential series

$$\exp_n(X) = 1 + X + \frac{X^2}{2!} + \cdots + \frac{X^n}{n!}$$

is an irreducible polynomial of $\mathbb{Q}[X]$.

$$\mathrm{Exp}_n(X) = n! \times \exp_n(X) = X^n + \sum_{k=0}^{n-1} \frac{n!}{k!} X^k \quad \text{is irreducible in } \mathbb{Z}[X].$$

Schur's generalization:

$$1 + c_1 X + c_2 \frac{X^2}{2!} + \cdots + c_{n-1} \frac{X^{n-1}}{(n-1)!} + \frac{X^n}{n!} \quad \text{where } c_i \in \mathbb{Z}\ (1 \leq i \leq n-1)$$

is irreducible in $\mathbb{Q}[X]$.

Fares' Suggestion

In the late 1990s, Bhargava associated to any infinite subset E of $\mathbb{Z}$ generalized factorials denoted by $\{k!_E\}_{k\geq 0}$ and suggested to consider:

$$\exp_E(X) = 1 + \frac{X}{1!_E}X + \frac{X^2}{2!_E} + \cdots + \frac{X^k}{k!_E} + \cdots$$

$[\forall k \; k!|k!_E\,] \;\Rightarrow\; \exp_E$ is an entire function.

Fares' suggestion: to extend Schur's result to this generalized exponential.

Analogously, we consider the truncated exponential polynomials

$$\exp_{E,n}(X) = 1 + \frac{X}{1!_E}X + \frac{X^2}{2!_E} + \cdots + \frac{X^n}{n!_E},$$

as well as

$$\mathrm{Exp}_{E,n}(X) = n!_E \times \exp_{E,n}(X) = X^n + \sum_{k=0}^{n-1} \frac{n!_E}{k!_E} X^k$$

$[\forall k \leq n \quad k!_E|n!_E\,] \;\Rightarrow\; \mathrm{Exp}_{E,n} \in \mathbb{Z}[X].$

About Bhargava's factorials

Definition (Fix an infinite subset E of $\mathbb{Z}$ and consider)

$*$the ring $\mathrm{Int}(E,\mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(E) \subseteq \mathbb{Z}\}$ of integer-valued poly. on E,
$*$the sets $\mathfrak{I}_n(E,\mathbb{Z})$ of leading coefficients of the polynomials of $\mathrm{Int}_n(E,\mathbb{Z})$,
$*$the positive generators $\frac{1}{n!_E}$ of the fractional ideals $\mathfrak{I}_n(E,\mathbb{Z})$,
Their inverses are *Bhargava's factorials* $n!_E$ associated to E.

Lemma

1- *If* $E \subseteq F$, *then* $n!_F$ *divides* $n!_E$ *for every* $n \in \mathbb{N}$.
2- *For every* $n \in \mathbb{N}$, $n!$ *divides* $n!_E$.
3- *For* $0 \leq k \leq n$, $k!_E$ *divides* $n!_E$.
4- *For every* $n, m \in \mathbb{N}$, $n!_E \cdot m!_E$ *divides* $(n+m)!_E$.

Proof. 1- $E \subseteq F \Rightarrow \mathrm{Int}(F,\mathbb{Z}) \subseteq \mathrm{Int}(E,\mathbb{Z}) \Rightarrow \frac{1}{n!_F}\mathbb{Z} \subseteq \frac{1}{n!_E}\mathbb{Z} \Rightarrow n!_F | n!_E$.
3- $\mathfrak{I}_k(E,\mathbb{Z}) \subseteq \mathfrak{I}_n(E,\mathbb{Z}) \Rightarrow \frac{1}{k!_E}\mathbb{Z} \subseteq \frac{1}{n!_E}\mathbb{Z} \Rightarrow \frac{n!_E}{k!_E} \in \mathbb{Z}$.
4- $\mathrm{Int}_n(E,\mathbb{Z}) \cdot \mathrm{Int}_m(E,\mathbb{Z}) \subseteq \mathrm{Int}_{n+m}(E,\mathbb{Z})$.

Extension of Schur's arguments

By contradiction: $\mathrm{Exp}_{E,n}(X)$ reducible $\Rightarrow \exists\, f(X) \mid \mathrm{Exp}_{E,n}(X)$ such that $f \in \mathbb{Z}[X]$ is monic, irreducible, and with degree $r \leq \lfloor \frac{n}{2} \rfloor$.

Lemma (First argument)

If $f(X)$ divides $\mathrm{Exp}_{E,n}(X)$ in $\mathbb{Z}[X]$, is monic and of degree r, then $f(0)$ is divisible by every prime divisor p of $\frac{n!_E}{(n-r)!_E}$.

Proof.

Let p be a prime divisor of $\frac{n!_E}{(n-r)!_E}$.

Then, p divides $\frac{n!_E}{i!_E}$ for $0 \leq i \leq n-r$.

Thus, $\mathrm{Exp}_{E,n}(X)$ mod p is divisible by X^{n-r+1}.

As $\deg\left(\frac{\mathrm{Exp}_{E,n}(X)}{f(X)}\right) = n-r < n-r+1$, X divides $f(X)$ modulo p. □

Lemma (Second Argument)

If $f(X)$ divides $\mathrm{Exp}_{E,n}(X)$ in $\mathbb{Z}[X]$, is monic, of degree r, and irreducible, and if $p|f(0)$, then there exists $k \in \{1, \dots, n\}$ such that $r \geq \frac{k}{v_p(k!_E)}$.

Proof.

Let α be a root of f, $K = \mathbb{Q}(\alpha)$, $\mathcal{O}_K$ the ring of integers of K.

$p|f(0)$ and $f(0) = \pm N_{K/\mathbb{Q}}(\alpha) \;\Rightarrow\; \exists \mathfrak{q} \in \mathrm{Spec}(\mathcal{O}_K)$ s.t. $p \in \mathfrak{q}$ and $\alpha \in \mathfrak{q}$

$\mathrm{Exp}_{E,n}(\alpha) = 0 \Rightarrow -n!_E = \sum_{k=1}^{n} \frac{n!_E}{k!_E}\alpha^k \Rightarrow v_\mathfrak{q}(n!_E) \geq \min_k v_\mathfrak{q}\left(\frac{n!_E}{k!_E}\alpha^k\right)$

$$\Rightarrow \exists k \text{ such that } \quad k v_\mathfrak{q}(\alpha) \leq v_\mathfrak{q}(k!_E).$$

$$v_\mathfrak{q}(\alpha) \geq 1 \quad \text{and} \quad v_\mathfrak{q}(k!_E) = e(\mathfrak{q}/p)\, v_p(k!_E) \leq r \times v_p(k!_E)$$

$$k \leq k v_\mathfrak{q}(\alpha) \leq v_\mathfrak{q}(k!_E) \leq r \times v_p(k!_E).$$

□

Recall that the *p-valuative capacity* of a subset E of $\mathbb{Z}$ is defined by

$$\delta_p(E) = \lim_{k \to +\infty} \frac{v_p(k!_E)}{k}$$

and that $\delta_p(E) = \sup_{k \geq 1} \frac{v_p(k!_E)}{k}$ while $\delta_p(E)$ is never a maximum.

Corollary (of the second argument)

If $f(X)$ divides $\mathrm{Exp}_{E,n}(X)$ in $\mathbb{Z}[X]$, is monic, of degree r, and irreducible, and if $p | f(0)$, then

$$r > \frac{1}{\delta_p(E)}.$$

Theorem (Third Argument: Sylvester's theorem)

The product of s consecutive integers $> s$ is divisible by some $p > s$.

Proof of Schur's theorem.

By the third argument, since $n \geq 2(n-r)$, there exists $p > n - r \geq r$ which divides $\frac{n!}{(n-r)!}$.

By the first argument, p divides $f(0)$.

By the second argument, $r > \frac{1}{\delta_p(\mathbb{Z})}$ where $\delta_p(\mathbb{Z}) = \lim_k \frac{v_p(k!)}{k} = \frac{1}{p-1}$ since

$$v_p(k!) = \sum_{h \geq 1} \left\lfloor \frac{k}{p^h} \right\rfloor = \left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{k}{p^2} \right\rfloor + \cdots \quad [\textit{Legendre}]$$

Consequently, $r > p - 1$, that is, $r \geq p$ in contradiction with $p > r$.

$f(X)$ cannot exist and $\mathrm{Exp}_n(X)$ is irreducible. □

Looking for infinite subsets E of $\mathbb{Z}$ such that $\mathrm{Exp}_{E,n}(X)$ is irreducible for all or for some n

Example ($\mathbf{E = a\,\mathbb{N}}$ where $\mathbf{a \in \mathbb{N}^*}$)

$$n!_{a\mathbb{N}} = a^n\, n! \qquad \exp_{a\mathbb{N}}(X) = \exp\left(\frac{X}{a}\right)$$

$$\forall n \geq 1 \qquad exp_{a\mathbb{N},n}(X) = \sum_{k=0}^{n} \frac{1}{a^k k!} X^k \quad \text{is irreducible.}$$

Example ($\mathbf{E = \mathbb{N}^{(2)} = \{n^2 \mid n \in \mathbb{N}\}}$)

$$n!_{\mathbb{N}^{(2)}} = \frac{(2n)!}{2} \qquad \delta_p(\mathbb{N}^{(2)}) = \frac{2}{p-1}$$

$$\forall n \geq 1 \qquad exp_{\mathbb{N}^{(2)},n}(X) = 1 + 2\sum_{k=1}^{n} \frac{X^k}{(2k)!} \quad \text{is irreducible.}$$

Proof.

$\frac{n!_{\mathbb{N}^{(2)}}}{(n-r)!_{\mathbb{N}^{(2)}}} = (2(n-r)+1)\cdots(2n-1)(2n) \Rightarrow \exists\, p > 2r$ such that $p | f(0)$.

$p|f(0) \;\Rightarrow\; \delta_p(\mathbb{N}^{(2)}) = \frac{2}{p-1} > \frac{1}{r} \Rightarrow p \leq 2r$. This is a contradiction. □

Analogously,

Example ($\mathbf{E = T = \left\{ \frac{n(n+1)}{2} \,\middle|\, n \geq 0 \right\}}$)

$$n!_T = \frac{(2n)!}{2^n} \qquad \delta_p(T) = \frac{2}{p-1} - v_p(2)$$

$$\forall n \geq 1 \qquad exp_{T,n}(X) = \sum_{k=0}^{n} \frac{2^k}{(2k)!} X^k \text{ is irreducible.}$$

An Obvious Generalization

If $F = \alpha E + \beta = \{\alpha x + \beta \mid x \in E\}$, then $n!_F = \alpha^n n!_E$ for every n.

Consequently,

$$\exp_F(X) = \exp_E\left(\frac{X}{\alpha}\right).$$

If Schur's theorem extends to E, then it also extends to $F = \alpha E + \beta$.

The case $E = \mathbb{P}$ (prime numbers)

$$v_p(n!_{\mathbb{P}}) = \sum_{k \geq 0} \left\lfloor \frac{n-1}{(p-1)p^k} \right\rfloor \quad , \quad n!_{\mathbb{P}} = \prod_{p \in \mathbb{P}} p^{\sum_{k \geq 0} \left\lfloor \frac{n-1}{(p-1)p^k} \right\rfloor}$$

$$\delta_p(\mathbb{P}) = \lim_{n \to +\infty} \frac{1}{n} \times \frac{n-1}{p-1} \times \sum_{k \geq 0} \frac{1}{p^k} = \frac{p}{(p-1)^2}.$$

Assume that:

A monic irreducible polynomial $f \in \mathbb{Z}[X]$ of degree r divides $\mathrm{Exp}_{\mathbb{P},n}(X)$.

Lemma (second argument: $p \mid f(0) \Rightarrow p \leq r+1$)

Proof.

$p|f(0) \Rightarrow \frac{(p-1)^2}{p} < r$, that is, $p + \frac{1}{p} < r + 2$ or $p \leq r + 1$. □

Looking for some $p \geq r+2$ dividing $f(0)$ to have a contradiction.

Lemma ($p \in \mathbb{P}$ and $r \in \mathbb{N}^*$ are such that $p \geq r+2$)

For $n \geq 2r$, $p \,\Big|\, \frac{n!_{\mathbb{P}}}{(n-r)!_{\mathbb{P}}}$ iff $n-1 = (p-1)q + s$ with $q \in \mathbb{N}^*$ and $0 \leq s < r$.

Sketch of the proof.

$$v_p\left(\frac{n!_{\mathbb{P}}}{(n-r)!_{\mathbb{P}}}\right) = \sum_{k\geq 0}\left\lfloor\frac{n-1}{(p-1)p^k}\right\rfloor - \sum_{k\geq 0}\left\lfloor\frac{n-r-1}{(p-1)p^k}\right\rfloor$$

$$p \,\Big|\, \frac{n!_{\mathbb{P}}}{(n-r)!_{\mathbb{P}}} \quad \Leftrightarrow \quad \left\lfloor\frac{n-1}{p-1}\right\rfloor - \left\lfloor\frac{n-1-r}{p-1}\right\rfloor > 0$$

$$p-1 > r \Rightarrow [\,(>0) \Leftrightarrow (=1)\,]$$

□

Theorem ($n \geq 2r > 0, p \geq r+2$)

If $n-1 = (p-1)q + s$ where $q \in \mathbb{N}^*$ and $0 \leq s < r$,
then $\exp_{\mathbb{P},n}(X)$ has no irreducible divisor of degree r.

Lemma (Roots)

If $\alpha \in \mathbb{Q}$ is a root of $\mathrm{Exp}_{E,n}(X)$, then $\alpha \in \mathbb{Z} \setminus \mathbb{N}$ and $v_p(\alpha) < \delta_p(E)\ \forall p$

Proof: $\mathrm{Exp}_{E,n}(\alpha) = 0$.

$\Leftrightarrow n!_E = \sum_{k=1}^{n} \frac{n!_E}{k!_E} \alpha^k \Rightarrow \forall p\, \exists k$ s.t. $v_p(\alpha) \leq \frac{v_p(k!_E)}{k} < \delta_p(E)$ □

Theorem

For all $n \geq 2$, the polynomial $\mathrm{Exp}_{\mathbb{P},n}(X)$ does not have any root in $\mathbb{Z}$.

Proof: Assume that $\alpha \in \mathbb{Z}$ is a root of $\mathrm{Exp}_{\mathbb{P},n}(X)$.

$\delta_p(\mathbb{P}) = \frac{p}{(p-1)^2} < 1$ for $p \neq 2$ and $\delta_2(\mathbb{P}) = 2 \Rightarrow \alpha \in \{-1, -2\}$. □

Thus, the inequalities to consider are

$$2 \leq r \leq \lfloor \tfrac{n}{2} \rfloor \text{ with } \begin{cases} p \geq r + 2 \\ 0 \leq s < r \end{cases}$$

Lemma (let $p \geq \lfloor \frac{n}{2} \rfloor + 2$)

$$p \mid \frac{n!_{\mathbb{P}}}{(n-r)!_{\mathbb{P}}} \text{ for all } r \text{ such that } 2 \leq r \leq \lfloor \tfrac{n}{2} \rfloor \iff n = p \text{ or } p+1.$$

Proof.

$[p \geq \lfloor \frac{n}{2} \rfloor + 2] \Rightarrow [n - 1 \geq \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) q + s > \frac{n}{2} q] \Rightarrow [q = 1].$ □

Corollary

For every $p \in \mathbb{P}$, $\exp_{\mathbb{P},p}(X)$ and $\exp_{\mathbb{P},p+1}(X)$ are irreducible.

Proof: $p \geq 5 \Rightarrow p \geq \left\lfloor \frac{p+1}{2} \right\rfloor + 2$; for $n = 2, 3, 4$ OK.

□

Example

For $1 \leq n \leq 8$, $\exp_{\mathbb{P},n}$ is irreducible.

The conditions that we used are only sufficient conditions

Note that, for a fixed n, we can consider several p. For instance:

Example

$\exp_{\mathbb{P},9}$ is irreducible. Consequently, $\exp_{\mathbb{P},n}$ is irreducible for $n \leq 14$.

Theorem

In fact, we have better results since there are several improvements of Sylvester's theorem.

Remark.

There are many interesting subsets E of $\mathbb{Z}$ for which it should be possible to prove the irreducibility of $\exp_{E,n}(X)$. □

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