Formal Unique Factorization Domains

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This talk is based on the following joint works with H.S. Choi.

1. G.W. Chang and H.S. Choi, *Almost Dedekind domains constructed from chains of Dedekind domains*, submitted.
2. ____, *Formal unique factorization domains*, submitted.

1. Motivation
2. Definition of a formal UFD
3. Three examples of a formal UFD

Motivation

The ring $D = \bigcup_{n\in\mathbb{N}_0} F[X^{\frac{1}{p^n}}, X^{-\frac{1}{p^n}}]$ for a prime number p

Let p be a prime number, F a field, X an indeterminate over F, and $D_n = F[X^{\frac{1}{p^n}}, X^{-\frac{1}{p^n}}]$ for each $n \in \mathbb{N}_0$. Then

- $D_n \cong F[y, y^{-1}]$ for an indeterminate y over F,
- D_n is a PID, and
- D_{n+1} is an integral extension of D_n for all $n \in \mathbb{N}_0$,

Theorem 1

Let $D = \bigcup_{n\in\mathbb{N}_0} D_n$. Then the following conditions hold.

1. $D = \bigcup_{n\in\mathbb{N}_0, n\geq k} D_n$ *for all* $k \in \mathbb{N}_0$.
2. D *is a one-dimensional Bezout domain.*
3. D *is an almost Dedekind domain if and only if char*$(F) \neq p$, *if and only if* D *is an SP domain.*

Prime elements of $D = \bigcup_{n \in \mathbb{N}_0} F[X^{\frac{1}{p^n}}, X^{-\frac{1}{p^n}}]$ I

Let $f(X) \in F[X]$ be an irreducible polynomial in $F[X]$ with $f(0) \neq 0$.

Lemma 2

Let $n \in \mathbb{N}_0$. Then $f(X)$ is irreducible in $F[X^{\frac{1}{p^n}}]$ if and only if $f(X^{p^n})$ is irreducible in $F[X]$.

Proposition 3

Let $D = \bigcup_{n \in \mathbb{N}_0} F[X^{\frac{1}{p^n}}, X^{-\frac{1}{p^n}}]$. Then the following statements are equivalent.

1. *$f(X)$ is an irreducible element of D.*
2. *If $p = 2$ (resp., $p > 2$), then $f(X^4)$ (resp., $f(X^p)$) is an irreducible element of $F[X]$.*
3. *$f(X^{p^n})$ is an irreducible element of $F[X]$ for all $n \in \mathbb{N}_0$.*

Example 4

Let $f(X) = a_0 + a_1X + \cdots + a_mX^m$ be a nonconstant polynomial of $\mathbb{Z}[X]$ such that $q \mid a_0, q \mid a_1, \ldots, q \mid a_{m-1}, q \nmid a_m, q^2 \nmid a_0$ for some prime number q. Then the following statements hold.

- $f(X^{p^n})$ is irreducible in $\mathbb{Q}[X]$ for all $n \in \mathbb{N}_0$ by Eisenstein's criterion.
- $f(X)$ is prime in D with $F = \mathbb{Q}$ by Proposition 3.

Example 5

Let $D = \bigcup_{n\in\mathbb{N}_0} \mathbb{Q}[X^{\frac{1}{2^n}}, X^{-\frac{1}{2^n}}]$. Note that $X^4 + 1$ is irreducible in $\mathbb{Q}[X]$ but $X^4 - 1$ is not irreducible in $\mathbb{Q}[X]$. Hence, by Proposition 3, $X + 1$ is a prime element of D but $X - 1$ is not a prime element of D. In particular, $X^{\frac{1}{2^n}} + 1$ is a prime element of D for all $n \in \mathbb{N}_0$.

Formal product of infinitely many prime elements

Notice that in $D = \bigcup_{n \in \mathbb{N}_0} \mathbb{Q}[X^{\frac{1}{2^n}}, X^{-\frac{1}{2^n}}]$, we have

$$\begin{aligned} X - 1 &= (X^{\frac{1}{2}} - 1)(X^{\frac{1}{2}} + 1) \\ &= (X^{\frac{1}{4}} - 1)(X^{\frac{1}{4}} + 1)(X^{\frac{1}{2}} + 1) \\ &= (X^{\frac{1}{8}} - 1)(X^{\frac{1}{8}} + 1)(X^{\frac{1}{4}} + 1)(X^{\frac{1}{2}} + 1) \\ &\vdots \\ &= (X^{\frac{1}{2^n}} - 1)\prod_{i=1}^{n}(X^{\frac{1}{2^i}} + 1) \\ &\vdots \end{aligned}$$

Thus, we want to write $X - 1 = \prod_{i=1}^{\infty}(X^{\frac{1}{2^i}} + 1)$.

Definition of a formal UFD

Definition of formal UFDs I

Let D be an integral domain and $a, b \in D$ be nonzero.

1. $a \mid b$ denotes $b = ad$ for some $d \in D$.
2. For $n \in \mathbb{N}_0$, $a^n \parallel b$ denotes that $a^n \mid b$ but $a^{n+1} \nmid b$.
3. $U(D)$ is the group of units in D.

Notation 6

(1) *Let S be a set of prime elements of D such that if $p \in D$ is a prime element, there exists a unique prime $q \in S$ so that $pD = qD$.*

(2) *Let $\langle S \rangle$ be the saturated multiplicative subset of D generated by S and $xU(D) = \{xu \mid u \in U(D)\}$ for all $x \in D$. Then, by the axiom of choice, there exists a subset T of $D \setminus (\langle S \rangle \cup \{0\})$ such that $|T \cap aU(D)| = 1$ for all $a \in D \setminus (\langle S \rangle \cup \{0\})$.*

Definition of formal UFDs II

Notation 7

(3) *Let* $g \in T$ *be such that there exists a unique set* $\{k_p \in \mathbb{N}_0 \mid p \in S\}$ *with the following properties:*

(a) $p^{k_p} \parallel g$ *for each* $p \in S$ *and*

(b) *if* $h \in D$ *is such that* $p^{k_p} \parallel h$ *for each* $p \in S$*, then* $g \mid h$.

Then we will write $g = \prod_{p \in S} p^{k_p}$.

(4) *If* $f \in D$ *is a nonzero nonunit satisfying (a) and (b) above, then there exists a unique unit* u_f *of* D *such that* $f = u_f \prod_{p \in S} p^{k_p}$*, which will be said to be a formal product of prime elements.*

Definition of formal UFDs III

Definition 8

An integral domain D is a formal UFD if

1. D is a GCD domain and
2. each nonzero nonunit can be written as a formal product of (possibly infinitely many) prime elements.

The following example seems to be correct, so the second condition of Definition 8 does not imply that D is a GCD domain.

Example 9 (M.H. Park's talk on Thursday)

Let V be a rank-one nondiscrete valuation domain and $V[[X]]$ be the power series ring over V.

1. $V[[X]]_{V\setminus\{0\}}$ satisfies the second condition of Definition 8.
2. $V[[X]]_{V\setminus\{0\}}$ need not be a GCD domain.

Three examples of a formal UFD

Example of formal UFDs I

Example 10

Let $D = \bigcup_{n\in\mathbb{N}_0} \mathbb{Q}[X^{\frac{1}{2^n}}, X^{-\frac{1}{2^n}}]$. Then D is a formal UFD.

Proof.

(1) D is a one-dimensional Bezout domain.

(2) Let $f(X) \in \mathbb{Q}[X]$ be a nonconstant polynomial with $f(0) \neq 0$. If $f(X)$ is irreducible over $\mathbb{Q}$, then the following are equivalent.

  (i) $f(X)$ is not a product of finitely many prime elements of D.

  (ii) $f(X) = u\Phi_d(X)$ for some nonzero $u \in \mathbb{Q}$ and an odd integer $d \in \mathbb{N}$.

(3) $d \in \mathbb{N}$ is even if and only if $\Phi_d(X)$ is a prime element of D.

(4) Let $k_1, \ldots, k_a$ be positive integers and $l_1, \ldots, l_a$ be distinct positive odd integers. Then $\Phi_{l_1}(X)^{k_1} \cdots \Phi_{l_a}(X)^{k_a} = \prod_{\substack{n\in\mathbb{N},\\ j\in\{1,\ldots,a\}}} \Phi_{2l_j}(X^{\frac{1}{2^n}})^{k_j}$.

□

Example of formal UFDs I cont.

Proof.

(5) Each nonzero nonunit of D can be written uniquely as a formal product of prime elements of D.
(Proof. Let S be the set of monic polynomials of $R = \bigcup_{n \in \mathbb{N}_0} \mathbb{Q}[X^{\frac{1}{2^n}}]$ that are prime elements of D.

1. If $f \in D$ is a nonzero nonunit, then $X^m f \in \mathbb{Q}[X^{\frac{1}{2^k}}]$ for some integers $m, k \in \mathbb{N}_0$, so we may assume that $f \in \mathbb{Q}[X^{\frac{1}{2^k}}]$.
2. $D = \bigcup_{n \in \mathbb{N}_0, n \geq k} \mathbb{Q}[X^{\frac{1}{2^n}}, X^{-\frac{1}{2^n}}]$, so we may assume that $k = 0$, and hence $f \in \mathbb{Q}[X]$ and $f(0) \neq 0$.
3. $f = f_1^{e_1} \cdots f_s^{e_s}$ is a prime (or an irreducible) factorization of f in $\mathbb{Q}[X]$.
4. Each f_i for $i = 1, \ldots, s$, and hence f can be written as a formal product of prime elements by (2), (3) and (4).
5. D is a Bezout domain by (1), so D is a formal UFD.)

□

Example of formal UFDs II

Example 11

Let D be an almost Dedekind Bézout domain and N be the saturated multiplicative subset of D generated by prime elements. If all but finitely many maximal ideals of D are principal, then D is a formal UFD.

Let $c(f)$ be the ideal of D generated by the coefficients of a polynomial $f \in D[X]$ and $D(X) = \{\frac{f}{g} \mid f, g \in D[X] \text{ and } c(g) = D\}$. Then $D(X)$, called the *Nagata ring* of D, is an overring of $D[X]$.

Corollary 12

Let D be an almost Dedekind domain. If all but finitely many of maximal ideals of D are invertible, then $D(X)$ is a formal UFD.

Example of formal UFDs III

The ring of entire functions 1

Example 13

Let E be the ring of entire functions. Then E is a formal UFD.

Proposition 14

Let $s_\alpha = z - \alpha$ for all $\alpha \in \mathbb{C}$, N be the saturated multiplicative subset of E generated by $\{s_\alpha \mid \alpha \in \mathbb{C}\}$ and D be an overring of E_N that is not a field. Then the following statements hold.

1. *$s_\alpha E$ is a height-one maximal ideal of E for all $\alpha \in \mathbb{C}$.*
2. $E = \bigcap_{\alpha \in \mathbb{C}} E_{s_\alpha E}$.
3. *D is not completely integrally closed.*
4. *The Krull dimension of D, denoted by $\dim(D)$, is uncountable.*
5. *D is not a formal UFD. In particular, E_N is not a formal UFD.*

For a nonzero $f \in E$, let $Z(f) = \{\alpha \in \mathbb{C} \mid f(\alpha) = 0\}$, so $Z(f)$ has no limit point in $\mathbb{C}$. Conversely, if $\mathcal{Z} = \{z_i\}_{i \in \mathbb{N}_0}$ is a discrete subset of $\mathbb{C}$ with no limit point in $\mathbb{C}$ and if $\{n_i\}_{i \in \mathbb{N}_0}$ is a sequence of positive integers, then there exists $f \in E$ such that $Z(f) = \mathcal{Z}$ and $\mathcal{O}_{z_i}(f) = n_i$ for each $i \in \mathbb{N}_0$.

Corollary 15

Let D be an overring of E that is not a field. The following are equivalent.

1. *D is a formal UFD.*
2. *D is completely integrally closed.*
3. *$D = \bigcap_{\alpha \in \mathcal{A}} E_{s_\alpha E}$ for a nonempty subset $\mathcal{A}$ of $\mathbb{C}$.*
4. *$D = E_N$, where $N = E \setminus (\bigcup_{\alpha \in \mathcal{A}} s_\alpha E)$ for a nonempty subset $\mathcal{A}$ of $\mathbb{C}$.*
5. *$D = E_N$, where $N = \{f \in E \mid Z(f) \cap \mathcal{A} = \emptyset\}$ for some $\emptyset \neq \mathcal{A} \subseteq \mathbb{C}$.*

Example of formal UFDs III

The ring of entire functions 3

Corollary 16

Let D be an overring of E that is not a field. The following are equivalent.

1. *D is a UFD, equivalently, a Dedekind domain or a PID.*
2. *D is an almost Dedekind domain.*
3. $dim(D) = 1$.
4. $dim(D) < \infty$.
5. $D = \bigcap_{\alpha \in \mathcal{A}} E_{s_\alpha E}$, *which is of finite character, for a nonempty $\mathcal{A} \subseteq \mathbb{C}$.*
6. $D = \bigcap_{\alpha \in \mathcal{A}} E_{s_\alpha E}$ *for a nonempty bounded subset $\mathcal{A}$ of $\mathbb{C}$.*
7. $D = E_N$, *where* $N = E \setminus (\bigcup_{\alpha \in \mathcal{A}} s_\alpha E)$ *or* $N = \{f \in E \mid Z(f) \cap \mathcal{A} = \emptyset\}$ *for a nonempty bounded subset $\mathcal{A}$ of $\mathbb{C}$.*

[0] E.M. Pirtle, *A note on overrings of the ring of entire functions*, Monatsh. Math. 75 (1971), 163-167.

A characterization of formal UFDs and Question

Theorem 17

Let $X_p^1(D)$ be the set of height-one principal prime ideals of a GCD domain D. Then the following statements are equivalent.

1. *D is a formal UFD.*
2. $D = \bigcap_{P \in X_p^1(D)} D_P$.
3. *$D[X]$, the polynomial ring over D, is a formal UFD.*

Question 18

Let D be an integral domain in which each nonzero nonunit can be written uniquely as a product of (possibly infinitely many) prime elements. Is D a GCD domain or a formal UFD ?

Thank you for your attention