Conference on Rings and Polynomials
TU Graz, Graz, Austria, July 14 – 19, 2025

Central polynomials of minimal degree for matrices

Vesselin Drensky[1]
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
drensky@math.bas.bg

July 15, 2025

[1]This is a joint work with Boyan Kostadinov whose research was partially supported by the National Programme of Ministry of Education and Science of Bulgaria "Young Scientists and Post-Doctoral Researchers - 2".

Central polynomials

Let K be a field of characteristic 0 and let $M_n(K)$ be the $n \times n$ matrix algebra over K. (Some of the results in the introduction part of the talk hold also over any field of arbitrary characteristic.)

By the Cayley-Hamilton theorem for 2×2 matrices

$$a^2 - \mathrm{tr}(a)a + \det(a)I = 0,\ a \in M_2(K),$$

where I is the identity matrix. Since $\mathrm{tr}(ab) = \mathrm{tr}(ba)$, $a, b \in M_2(K)$, we obtain that $\mathrm{tr}([a, b]) = 0$, where $[a, b] = ab - ba$ is the commutator of a, b. Hence

$$[a, b]^2 = -\det([a, b])I.$$

The polynomial $c(x_1, x_2) = [x_1, x_2]^2$ from the free associative algebra $K\langle X\rangle = K\langle x_1, x_2, \ldots\rangle$ has the property that evaluated on the algebra of 2×2 matrices produces scalar matrices only but does not vanish identically on $M_2(K)$ (i.e. is not a polynomial identity). Since its evaluations belong to the center of $M_2(K)$ such a polynomial is called central.

In 1956 Kaplansky gave a talk where he asked 12 problems which motivated significant research activity in the next decades. One of his problems was whether there exists a central polynomial for the matrix algebras $M_n(K)$ for $n > 2$.

- I. Kaplansky, Problems in the theory of rings, Report of a conference on linear algebras: June 6-8, 1956, Ram's Head Inn, Shelter Island, New York, National Academies Press, Washington, DC, 1957, 1-3.

Problem 5. Let A_n denote the n by n total matrix algebra over a field. Does there exist a polynomial which always takes values in the center of A_n without being identically 0?

For $n = 2$, $(xy - yx)^2$ is such a polynomial.

- I. Kaplansky, "Problems in the theory of rings" revisited, Am. Math. Mon. 77 (1970), 445-454.

The problem is stated a little carelessly. The polynomial 1 might be offered as an answer! Or, if this is too bizarre, we could add 1 to, say, the standard identity.

Herstein showed me (he credits the observation to John Thompson) that over a finite field one can construct a polynomial in one variable which sends every n by n matrix to a scalar, and not always the same scalar. This is again unsatisfactory. Let us revamp Problem 5, and give it a new number to avoid confusion. While we are at it let us insert the restriction $n \geqq 3$, for the problem is motivated by the polynomial $(xy-yx)^2$ which works so nicely for $n=2$. (Strictly speaking, $(xy-yx)^2$ should be linearized to meet the conditions of Problem 16.)

Problem 16. Let A_n denote the n by n total matrix algebra over a field, $n \geqq 3$. Does there exist a homogeneous multilinear polynomial (of positive degree) which always takes values in the center of A_n without being identically 0?

The fact that the matrices over a finite field have central polynomials in one variable was established by Latyshev and Shmelkin.

- V.N. Latyshev, A.L. Shmelkin, A certain problem of Kaplansky (Russian), Algebra i Logika 8 (1969), 447-448. Translation: Algebra and Logic 8 (1969), p. 257.

The answer to the problem of Kaplansky was given in 1972-1973 by Formanek and Razmyslov. Later, 1979 an alternative one-page proof of the existence of central polynomials based on results of Amitsur was given by Kharchenko.

- E. Formanek, Central polynomials for matrix rings, J. Algebra 23 (1972), 129-132.
- Yu.P. Razmyslov, On a problem of Kaplansky (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. 37 (1973), 483-501. Translation: Math. USSR, Izv. 7 (1973), 479-496.
- V.K. Kharchenko, A remark on central polynomials (Russian), Mat. Zametki 26 (1979), 345-346. Translation: Math. Notes 26 (1979), p. 665.

The existence of central polynomials for matrices was very important for ring theory. Very soon important theorems were established or simplified using central polynomials.

- N. Jacobson, PI-Algebras: An Introduction, Lecture Notes in Math. 441, Springer-Verlag, Berlin-New York, 1975.

The central polynomial of Formanek for $M_n(K)$ is of degree n^2 and the polynomial of Razmyslov is of degree $3n^2 - 1$.
Using the method of Razmyslov, Halpin constructed a central polynomial which is also of degree n^2.

- P. Halpin, Central and weak identities for matrices, Commun. in Algebra 11 (1983), 2237-2248.

Common belief

The minimal degree of the central polynomials of $M_n(K)$ is n^2.

This is true for $n \leq 2$.

Central polynomials of low degree

In 1984 Drensky and Azniv Kasparian constructed a central polynomial of degree 8 for $M_3(K)$ and showed that $M_3(K)$ does not have central polynomials of lower degree. The new central polynomial of degree 8 was obtained using ideas of the Rosset proof of Amitsur-Levitzki Theorem. (The proof of Rosset uses matrices with entries from the Grassmann algebra.) The approach to show that there are no central polynomials of degree 7 was based on computations by hand combined with techniques of representation theory of the general linear group. With the same method more central polynomials of degree 8 were found. These and other new results were included in the master thesis of Azniv Kasparian (with supervisor Drensky).

- A. Kasparian, Polynomial Identities of Matrices (Bulgarian), Master Thesis, University of Sofia, 1984.
- V. Drensky, A. Kasparian, Polynomial identities of eighth degree for 3×3 matrices, Annuaire de l'Univ. de Sofia, Fac. de Math. et Mecan., Livre 1, Math. 77 (1983), 175-195.
- V. Drensky, A. Kasparian, A new central polynomial for 3×3 matrices, Commun. in Algebra 13 (1985), 745-752.

Using a computer, Bondari found all central polynomials of degree 8 for $M_3(K)$. Three of them were know from the paper by Drensky and Kasparian and one of them was new.

- S. Bondari, Constructing the polynomial identities and central identities of degree < 9 of 3×3 matrices, Linear Algebra Appl. 258 (1997), 233-249.

The conjecture of Formanek

The minimal degree of the central polynomials for $M_n(K)$ over a field K of characteristic 0 is equal to

$$\mathrm{mindeg}(M_n(K)) = \frac{1}{2}(n^2 + 3n - 2).$$

- E. Formanek, The Polynomial Identities and Invariants of $n \times n$ Matrices, CBMS Regional Conf. Series in Math. 78, Published for the Confer. Board of the Math. Sci. Washington DC, AMS, Providence RI, 1991.

Although this is not a reason, the only quadratic function $p(n)$ with $p(1) = 1$, $p(2) = 4$ and $p(3) = 8$ is $\frac{1}{2}(n^2 + 3n - 2)$. What is more important, the conjecture of Formanek agrees with some other conjectures in the theory of PI-algebras.

Drensky and Tsetska Rashkova studied weak polynomial identities of matrices. (The knowledge of weak polynomial identities is a key moment of the construction of central polynomials by Razmyslov.) In particular the obtained by computer results explained the existence of a central polynomial of degree 8 for 3×3 matrices.

- V. Drensky, Ts.G. Rashkova, Weak polynomial identities for the matrix algebras, Commun. in Algebra 21 (1993), 3779-3795.

For $n = 4$ Drensky and Giulia Maria Piacentini Cattaneo found a new central polynomial of degree 13. (Pay attention: $\frac{1}{2}(4^2 + 3 \cdot 4 - 2) = 13$. Unfortunately we do not know whether $M_4(K)$ has central polynomials of degree 12.) The construction uses a weak polynomial identity of degree 9 found by computer (with methods similar to those of Drensky and Rashkova) and combines the methods of Formanek and Razmyslov. The result was generalized by Drensky to construct central polynomials of degree $(n-1)^2 + 4$ for $M_n(K)$, $n > 2$.

- V. Drensky, G.M. Piacentini Cattaneo, A central polynomial of low degree for 4×4 matrices, J. Algebra 168 (1974), 469-478.
- V. Drensky, New central polynomials for the matrix algebra, Israel J. Math. 92 (1995), 235-248.

Other central polynomials

Giambruno and Valenti constructed central polynomials using techniques from invariant theory of matrices.

- A. Giambruno, A. Valenti, Central polynomials and matrix invariants, Israel J. Math. 96 (1996), 281-297.

The proof of Kharchenko from 1979 is not constructive. There are several other nonconstructive proofs. The main ideas are similar. As the proof of Kharchenko, the proof of Braun in 1982 used old results by Amitsur from the 1950's, before Kaplansky stated his problem. The more recent proofs of Brešar are more self-contained and use techniques typical for generalized polynomial identities and functional identities.

- A. Braun, On Artin's theorem and Azumaya algebras, J. Algebra 77 (1982), 323-332.
- M. Brešar, An alternative approach to the structure theory of PI-rings, Expo. Math. 29 (2011), No. 1, 159-164.
- M. Brešar, A unified approach to the structure theory of PI-rings and GPI-rings, Serdica Math. J. 38 (2012), No. 1-3, 199-210.

Brešar and Drensky showed that starting with a multihomogeneous central polynomial for $M_n(K)$ when K is an infinite field of positive characteristic p one can produce a polynomial of the same degree with coefficients in the prime field $\mathbb{F}_p$ which is central for the $n \times n$ matrices over any field F of characteristic p. The proof is elementary and uses only standard combinatorial techniques. It also completes the nonconstructive proofs removing the requirement of the infinity of the field K in some of them.

- M. Brešar, V. Drensky, Central polynomials for matrices over finite fields, Linear Multilinear Algebra 61 (2013), No. 7, 939-944.

The conjecture of Formanek for $n = 4$

If we want to confirm the conjecture of Formanek, the first step is to show that there are no central polynomials of degree 12 for $M_4(K)$.

Naive and hopeless approach

It is sufficient to show that there are no multilinear central polynomials. Let

$$f(x_1,\dots,x_{12})=\sum_{\sigma\in S_{12}}\xi_\sigma x_{\sigma(1)}\cdots x_{\sigma(12)}$$

be the candidate for a central polynomial of degree 12 with $12!=479001600$ unknown coefficients $\xi_\sigma\in K$, $\sigma\in S_{12}$. We replace the variables x_i with all possible matrix units $e_{p_iq_i}$:

$$f(e_{p_1q_1},\dots,e_{p_{12}q_{12}})=\sum_{r,s=1}^{4}f_{rs}^{(p,q)}e_{rs},$$

where $f_{rs}^{(p,q)}$, $(p,q)=((p_1,q_1),\dots,(p_{12},q_{12}))$, are linear combinations of ξ_σ.

We consider the linear homogeneous system with 12! unknowns ξ_σ

$$f_{rs}^{(p,q)} = 0,\ r, s = 1, 2, 3, 4,\ (p, q) = ((p_1, q_1), \dots, (p_{12}, q_{12})).$$

The solutions of the system are all multilinear polynomial identities of degree 12 for $M_4(K)$. If we determine the rank of the system, this would give us the dimension of the vector space of the polynomial identities. Then consider the system which consists of the equations with $r \neq s$ and add the equations of the form

$$f_{11}^{(p,q)} = f_{22}^{(p,q)} = f_{33}^{(p,q)} = f_{44}^{(p,q)}.$$

The solutions of the system give the polynomial identities and the central polynomials of degree 12. If the ranks of both systems are equal, this means that there are no central polynomials.

Better approach

We can use representation theory of the symmetric group. The vector space P_{12} of the multilinear polynomials of degree 12 in $K\langle X\rangle$ is a left S_{12}-module isomorphic to the group algebra KS_{12} considered as a left S_{12}-module. For each partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of 12 we consider a linear homogeneous system with number of unknowns equal to the degree of the associated irreducible character χ_λ.

$[\lambda]$	$\deg \chi_\lambda$	$[\lambda]$	$\deg \chi_\lambda$	$[\lambda]$	$\deg \chi_\lambda$	$[\lambda]$	$\deg \chi_\lambda$
$[12]$	1	$[821^2]$	945	$[651]$	1155	$[543]$	2112
$[11, 1]$	11	$[81^4]$	330	$[642]$	2673	$[5421]$	5775
$[10, 2]$	54	$[75]$	297	$[641^2]$	3080	$[541^3]$	3520
$[10, 1^2]$	55	$[741]$	1408	$[63^2]$	1650	$[53^2 1]$	4158
$[93]$	154	$[732]$	1925	$[6321]$	5632	$[532^2]$	4455
$[921]$	320	$[731^2]$	2376	$[631^3]$	3696	$[4^3]$	462
$[91^3]$	165	$[72^2 1]$	2079	$[62^3]$	1925	$[4^2 31]$	2970
$[84]$	275	$[721^3]$	1728	$[62^2 1^2]$	3564	$^*[621^4]$	2100
$[831]$	891	$[71^5]$	462	$[5^2 2]$	1320	$^*[5321^2]$	7700
$[82^2]$	616	$[6^2]$	132	$[5^2 1^2]$	1485	$^*[4^2 2^2]$	2640
						$[1^{12}]$	1

Instead of linear combinations of monomials $x_{\sigma(1)}\cdots x_{\sigma(12)}$ we consider the polynomials in P_{12}

$$\sum_{i=1}^{\deg(\chi_\lambda)} \eta_i f_\lambda^{(i)}(x_1,\dots,x_{12})$$

where every linear combinations of the polynomials $f_\lambda^{(i)}$ generates an irreducible S_{12}-module. Then we repeat the method described for the linear combinations of monomials and compare the ranks of the homogeneous linear systems. The number of the unknowns is much smaller then $12!$ (≤ 7700) but the polynomials $f_\lambda^{(i)}$ are complicated.

The new result

Our result in this talk is the following.

Theorem. *The matrix algebra* $M_4(K)$ *does not have central polynomials in two variables of degree* 12.

More perspective naive approach

As in the case of multilinear polynomials, consider the polynomial $f(x_1, x_2)$ which is bihomogeneous of bidegree (λ_1, λ_2)

$$f(x_1, x_2) = \sum_i \vartheta_i x_{i_1} \cdots x_{i_{12}}.$$

The number of unknowns ϑ_i is $\binom{12}{\lambda_i}$ but we have to replace x_1, x_2 with generic 4×4 matrices

$$y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14} \\ y_{21} & y_{22} & y_{23} & y_{24} \\ y_{31} & y_{32} & y_{33} & y_{34} \\ y_{41} & y_{42} & y_{43} & y_{44} \end{pmatrix}, z = \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \\ z_{31} & z_{32} & z_{33} & z_{34} \\ z_{41} & z_{42} & z_{43} & z_{44} \end{pmatrix}.$$

Use representation theory of the general linear group!

In the theory of PI-algebras the results obtained in the language of representation theory of the general linear group can be easily restated in the language of representation theory of the symmetric group. The advantage is that the considered polynomials are simpler than in the case of the symmetric group.

In our case one considers bihomogeneous polynomials $w_\lambda(x_1, x_2)$ which are highest weight vectors of irreducible $GL_2(K)$-modules corresponding to the partition $\lambda = (\lambda_1, \lambda_2)$. Such polynomials are characterized by the property that $w_\lambda(x_1, x_2 + x_1) = w_\lambda(x_1, x_2)$.

The number of the unknowns is ≤ 297.

Our method

The proof is based on combinatorial theory of PI-algebras and representation theory of the general linear group combined with work with systems for algebraic computations. As in the computational approach of Drensky and Kasparian in the 1983-paper, a central polynomial of minimal degree should be a linear combination of products of ≥ 4 commutators

$$[x_2, x_1]\mathrm{ad}^{p_1}(x_1)\mathrm{ad}^{p_2}(x_2),$$

where $u\mathrm{ad}(v) = [u, v]$. An additional requirement is that the polynomials are linear combinations of highest weight vectors $w_m(x_1, x_2)$ of irreducible $GL_2(K)$-modules corresponding to the partition $\lambda = (\lambda_1, \lambda_2)$, $\lambda_1 \geq \lambda_2 \geq 4$.

It follows from the known facts on the polynomial identities of $M_4(K)$ that in our approach we have to show that there are no central polynomials in two variables of degree 10, 11 and 12. In each of the cases we consider the linear combination

$$w_\lambda(y,z)=\sum \xi_m w_m(y,z),$$

where y,z are generic matrices and the ξ_m's are unknowns. The condition that $w_\lambda(y,z)$ is a scalar matrix gives a linear homogeneous system. It has turned out that in all cases

$$\lambda=(6,4),(5,5),(7,4),(6,5),(8,4),(7,5),(6,6)$$

the system has only the trivial solution.

Using purely theoretical arguments, the case $\lambda_2 = 4$ is impossible because we proved the following general theorem.

Theorem. *If $c(x_1, x_2)$ is a bihomogeneous central polynomial of degree (m, n) for $M_n(K)$, then $\deg(c) \geq n^2$.*

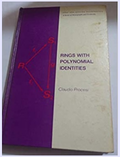
The most complicated cases are $\lambda = (7, 5)$ and $\lambda = (6, 6)$ when the system has 60 and 31 unknowns, respectively. In all the cases we have to apply various tricks to simplify the computations.

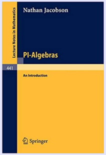
Conclusion

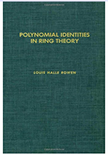
The theory of polynomial identities and central polynomials for matrices is an important part of the theory of PI-algebras.

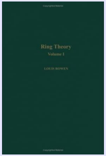
The class of PI-algebras is sufficiently big and contains important classes of algebras (e.g. all commutative and all finite dimensional algebras). On the other hand it is reasonably small and has rich structure and combinatorial theory.

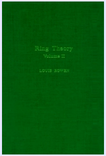
Further readings

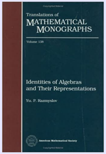


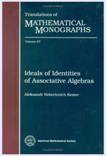


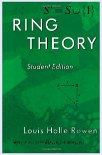


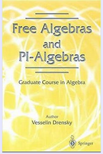


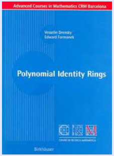


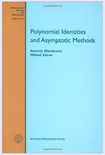


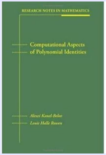


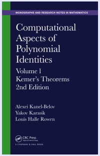


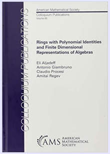












- C. Procesi, Rings with Polynomial Identities, Marcel Dekker, New York, 1973.
- N. Jacobson, PI-Algebras: An Introduction, Lecture Notes in Math. 441, Springer-Verlag, Berlin-New York, 1975.
- L.H. Rowen, Polynomial Identities of Ring Theory, Acad. Press, 1980.
- L.H. Rowen, Ring Theory, vol. 1, 2, Academic Press, Inc., Boston, MA, 1988.
- Yu.P. Razmyslov, Identities of Algebras and Their Representations (Russian), "Sovremennaya Algebra", "Nauka", Moscow, 1989. Translation: Translations of Math. Monographs 138, AMS, Providence, RI, 1994.

- A.R. Kemer, Ideals of Identities of Associative Algebras, Translations of Math. Monographs 87, AMS, Providence, RI, 1991.
- L.H. Rowen, Ring Theory, Student Edition, Academic Press, Inc., Boston, MA, 1991.
- V. Drensky, Free Algebras and PI-algebras. Graduate Course in Algebra, Singapore, Springer, 2000.
- V. Drensky, E. Formanek, Polynomial Identity Rings, Advanced Courses in Mathematics – CRM Barcelona, Basel, Birkhäuser, 2004.
- A. Giambruno, M. Zaicev, Polynomial Identities and Asymptotic Methods, Mathematical Surveys and Monographs 122, Providence, RI, American Mathematical Society, 2005.

- A. Kanel-Belov, L.H. Rowen, Computational Aspects of Polynomial Identities, Research Notes in Mathematics 9, Wellesley, MA, A K Peters, 2005.
- A. Kanel-Belov, Y. Karasik, L.H. Rowen, Computational Aspects of Polynomial Identities. Volume I: Kemers Theorems. 2nd edition, Monographs and Research Notes in Mathematics. Boca Raton, FL: CRC Press, 2016.
- E. Aljadeff, A. Giambruno, C. Procesi, A. Regev, Rings with Polynomial Identities and Finite Dimensional Representations of Algebras, Colloquium Publications. American Mathematical Society 66. Providence, RI, American Mathematical Society, 2020.

THANK YOU VERY MUCH FOR YOUR ATTENTION!