Translating M-Ideals: From Banach Space Geometry to Ring-Theoretic Structures

Amartya Goswami

Department of Mathematics and Applied Mathematics
University of Johannesburg

&

National Institute for Theoretical and Computational Sciences

? Who are the partners in this crime?

Who are the partners in this crime?

David P. Blecher (Houston University, USA)

? Who are the partners in this crime?

David P. Blecher (Houston University, USA)


Elena Caviglia   Luca Mesiti
(Post-doctoral fellows, Stellenbosch University, South Africa)

? What is the motivation?

? What is the motivation?

- M-ideals arise naturally in Banach space theory via duality and L-summands.

? What is the motivation?

- M-ideals arise naturally in Banach space theory via duality and L-summands.
- Our aim: to generalize this concept to purely algebraic settings, particularly to ring theory.

? What is the motivation?

- M-ideals arise naturally in Banach space theory via duality and L-summands.
- Our aim: to generalize this concept to purely algebraic settings, particularly to ring theory.
- Investigate whether similar structural properties persist in absence of topology.

? How it is started?

? How it is started?

 Blecher $\Rightarrow$ visited UJ (2024) $\Rightarrow$ Gave a talk

? How it is started?

Blecher ⇒ visited UJ (2024) ⇒ Gave a talk

E. M. Alfsen and E. G. Effros, Structure in real Banach spaces. Part I, *Ann. Math.*, **96**(1) (1972), 98–128.

? How it is started?

Blecher ⇒ visited UJ (2024) ⇒ Gave a talk

E. M. Alfsen and E. G. Effros, Structure in real Banach spaces. Part I, *Ann. Math.*, **96**(1) (1972), 98–128.

Suppose that V is a real Banach space, and that W is the dual Banach space of V. A subspace N of W is said to be an L-summand of W if there is a subspace M with $N \cap M = \{0\}$, $N + M = W$, and for $p \in N$, $q \in M$, $\|p+q\| = \|p\| + \|q\|$. A closed subspace J of V is said to be an $\mathscr{M}$-ideal if its annihilator $J^{\perp}$ is an L-summand in W.

? How it is started?

Blecher $\Rightarrow$ visited UJ (2024) $\Rightarrow$ Gave a talk

E. M. Alfsen and E. G. Effros, Structure in real Banach spaces. Part I, *Ann. Math.*, **96**(1) (1972), 98–128.

Suppose that V is a real Banach space, and that W is the dual Banach space of V. A subspace N of W is said to be an L-summand of W if there is a subspace M with $N \cap M = \{0\}$, $N + M = W$, and for $p \in N$, $q \in M$, $\|p+q\| = \|p\| + \|q\|$. A closed subspace J of V is said to be an $\underline{\mathscr{M}\text{-ideal}}$ if its annihilator $J^{\perp}$ is an L-summand in W.

(Theorem 5.8) An ideal J is an $\mathscr{M}$-ideal if and only if the following condition holds: If $B_1, \ldots, B_n$ are open balls with $B_1 \cap \cdots \cap B_n \neq \emptyset$ and $B_i \cap J \neq \emptyset$ for all i, then $B_1 \cap \cdots \cap B_n \cap J \neq \emptyset$.

? How to extend it to rings?

? How to extend it to rings?

An ideal J of a Banach space X is called an <u>$\mathscr{M}$-ideal</u> if for B_1, …, B_n open balls of X with $B_1 \cap \cdots \cap B_n \neq \emptyset$ and $B_i \cap J \neq \emptyset$ for all i, implies that

$$B_1 \cap \cdots \cap B_n \cap J \neq \emptyset.$$

How to extend it to rings?

An ideal J of a Banach space X is called an <u>$\mathscr{M}$-ideal</u> if for $B_1, \ldots, B_n$ open balls of X with $B_1 \cap \cdots \cap B_n \neq \emptyset$ and $B_i \cap J \neq \emptyset$ for all i, implies that

$$B_1 \cap \cdots \cap B_n \cap J \neq \emptyset.$$

? How to extend it to rings?

An ideal J of a Banach space X is called an $\mathscr{M}$-ideal if for $B_1, \ldots, B_n$ open balls of X with $B_1 \cap \cdots \cap B_n \neq \emptyset$ and $B_i \cap J \neq \emptyset$ for all i, implies that

$$B_1 \cap \cdots \cap B_n \cap J \neq \emptyset.$$

- Banach space $\Rightarrow$ Ring.

How to extend it to rings?

An ideal J of a Banach space X is called an $\mathscr{M}$-ideal if for $B_1, \ldots, B_n$ open balls of X with $B_1 \cap \cdots \cap B_n \neq \emptyset$ and $B_i \cap J \neq \emptyset$ for all i, implies that

$$B_1 \cap \cdots \cap B_n \cap J \neq \emptyset.$$

- Banach space $\Rightarrow$ Ring.
- Open balls $\Rightarrow$ Ideals.

? How to extend it to rings?

An ideal J of a Banach space X is called an $\mathscr{M}$-ideal if for B_1, …, B_n open balls of X with $B_1 \cap \cdots \cap B_n \neq \emptyset$ and $B_i \cap J \neq \emptyset$ for all i, implies that

$$B_1 \cap \cdots \cap B_n \cap J \neq \emptyset.$$

- Banach space $\Rightarrow$ Ring.
- Open balls $\Rightarrow$ Ideals.

An ideal J of a ring R is called an $\mathscr{M}$-ideal if for I_1, I_2, …, I_n ideals of R with $I_1 \cap \cdots \cap I_n \neq 0$ and $I_i \cap J \neq 0$ for each k for all i, implies that

$$I_1 \cap \cdots \cap I_n \cap J \neq 0.$$

? Can we reduce the number of balls and ideals?

? Can we reduce the number of balls and ideals?

⚖

Banach space $\mathscr{M}$-ideal	**Ring $\mathscr{M}$-ideal**
Sufficient to take three open balls, but *not* to two.	Sufficient to take two ideals.

Can we reduce the number of balls and ideals?

Banach space $\mathscr{M}$-ideal	**Ring $\mathscr{M}$-ideal**
Sufficient to take three open balls, but *not* to two.	Sufficient to take two ideals.

An integral commutative quantale (or multiplicative lattice) is a complete lattice $(L, \leqslant, 0, 1)$ endowed with an associative, commutative multiplication (denoted by $\cdot$), which distributes over arbitrary joins and has 1 as multiplicative identity.

Can we reduce the number of balls and ideals?

Banach space $\mathscr{M}$-ideal	Ring $\mathscr{M}$-ideal
Sufficient to take three open balls, but *not* to two.	Sufficient to take two ideals.

An integral commutative quantale (or multiplicative lattice) is a complete lattice $(L, \leqslant, 0, 1)$ endowed with an associative, commutative multiplication (denoted by $\cdot$), which distributes over arbitrary joins and has 1 as multiplicative identity.

An element x in a quantale L is called a $\mathscr{M}$-element if for every natural number $n \geqslant 2$ and every collection of n elements y_1, y_2, …, y_n in L such that $\bigwedge_{k=1}^{n} y_k \neq 0$, the condition $x \wedge y_k \neq 0$ for each k, implies that

$$x \wedge \left(\bigwedge_{k=1}^{n} y_k \right) \neq 0.$$

? Why (ring) $\mathscr{M}$-ideals are interesting?

Why (ring) $\mathscr{M}$-ideals are interesting?

An ideal J of a ring R is called <u>essential</u> if $J \cap I \neq 0$ for all nonzero ideals I of R.

? Why (ring) $\mathscr{M}$-ideals are interesting?

An ideal J of a ring R is called <u>essential</u> if $J \cap I \neq 0$ for all nonzero ideals I of R.

$\mathscr{M}$-ideals are a generalization of essential ideals.

? Why (ring) $\mathscr{M}$-ideals are interesting?

An ideal J of a ring R is called essential if $J \cap I \neq 0$ for all nonzero ideals I of R.

$\mathscr{M}$-ideals are a generalization of essential ideals.

Essential submodules play a pivotal role in the theory of injective modules via the notion of injective hulls*.

*The injective hull (or injective envelope) of a module is both the smallest injective module containing it and the largest *essential extension* of it

? Why (ring) $\mathscr{M}$-ideals are interesting?

An ideal J of a ring R is called essential if $J \cap I \neq 0$ for all nonzero ideals I of R.

$\mathscr{M}$-ideals are a generalization of essential ideals.

Essential submodules play a pivotal role in the theory of injective modules via the notion of injective hulls*.

*The injective hull (or injective envelope) of a module is both the smallest injective module containing it and the largest *essential extension* of it

$\mathscr{M}$-submodules: a possible generalization of injective hulls.

? Can we see some examples of $\mathscr{M}$-ideals?

? Can we see some examples of $\mathscr{M}$-ideals?

1. Essential ideals.

Can we see some examples of $\mathscr{M}$-ideals?

1. Essential ideals.
2. Zero ideal.

? Can we see some examples of $\mathscr{M}$-ideals?

1. Essential ideals.
2. ⚠ Zero ideal.
3. The maximal ideal of a nontrivial local ring ($\Leftarrow$ essential ideal, except field).

? Can we see some examples of $\mathscr{M}$-ideals?

1. Essential ideals.
2. Zero ideal.
3. The maximal ideal of a nontrivial local ring ($\Leftarrow$ essential ideal, except field).
4. Every ideal of an integral domain ($\Leftarrow$ essential ideal).

? Can we see some examples of $\mathscr{M}$-ideals?

1. Essential ideals.
2. ⚠ Zero ideal.
3. The maximal ideal of a nontrivial local ring ($\Leftarrow$ essential ideal, except field).
4. Every ideal of an integral domain ($\Leftarrow$ essential ideal).
5. ⚠ Every minimal ideal.

? Can we see some examples of $\mathscr{M}$-ideals?

1. Essential ideals.
2. Zero ideal.
3. The maximal ideal of a nontrivial local ring ($\Leftarrow$ essential ideal, except field).
4. Every ideal of an integral domain ($\Leftarrow$ essential ideal).
5. Every minimal ideal.

- $R := \mathbb{Z}_{30}$.
- $J := (2)$.
- $I_1 := (3)$ and $I_2 := (5)$.
- $(2) \cap (3) = (6)$, $(2) \cap (5) = (10)$, and $(3) \cap (5) = (15)$; but $(2) \cap (3) \cap (5) = 0$.

? Can we characterize $\mathscr{M}$-ideals?

? Can we characterize $\mathscr{M}$-ideals?

An ideal I of a ring R is an $\mathscr{M}$-ideal if and only if either it is essential or relatively irreducible*.

? Can we characterize $\mathscr{M}$-ideals?

An ideal I of a ring R is an $\mathscr{M}$-ideal if and only if either it is essential or relatively irreducible*.

*We say that an ideal I of a ring R is relatively irreducible if for every ideal J and K of R with $J \subseteq I$, $K \subseteq I$, and $J \cap K = 0$ implies that either $J = 0$ or $K = 0$.

? Consequences of the above characterization?

Consequences of the above characterization?

Every minimal ideal is relatively irreducible.

? Consequences of the above characterization?

Every minimal ideal is relatively irreducible.

Suppose n denotes the number of minimal ideals of a ring R. If $n \leqslant 1$, the $\mathrm{Soc}(R)$* is an $\mathscr{M}$-ideal. If $n \geqslant 2$, the $\mathrm{Soc}(R)$ is an $\mathscr{M}$-ideal if and only if it is essential.

*The sum of minimal ideals.

? Consequences of the above characterization?

Every minimal ideal is relatively irreducible.

Suppose n denotes the number of minimal ideals of a ring R. If $n \leqslant 1$, the $\mathrm{Soc}(R)$* is an $\mathscr{M}$-ideal. If $n \geqslant 2$, the $\mathrm{Soc}(R)$ is an $\mathscr{M}$-ideal if and only if it is essential.

*The sum of minimal ideals.

Consider a ring $\mathbb{Z}_n$ with $n = p_1^{m_1} \cdot \ldots \cdot p_k^{m_k}$. Then a nontrivial ideal

$$I := \left(p_1^{m'_1} \cdot \ldots \cdot p_k^{m'_k}\right)$$

is an $\mathscr{M}$-ideal if and only if there are not $i, j, s \in \{1, \ldots, k\}$ such that $i \neq j$, $m'_i < m_i$, $m'_j < m_j$, and $m'_s = m_s$. In particular, I is essential if and only if for every i we have $m'_i < m_i$.

? Further consequences?

? Further consequences?

Let X be a topological space and let $\mathsf{Open}(X)$ be the frame of open subsets of X. An open $U \in \mathsf{Open}(X)$ is a $\mathscr{M}$-element if and only if U is dense or irreducible. In particular, U is essential in $\mathsf{Open}(X)$ if and only if U is dense in X.

? Further consequences?

Let X be a topological space and let $\mathsf{Open}(X)$ be the frame of open subsets of X. An open $U \in \mathsf{Open}(X)$ is a $\mathscr{M}$-element if and only if U is dense or irreducible. In particular, U is essential in $\mathsf{Open}(X)$ if and only if U is dense in X.

The nonzero (ring) $\mathscr{M}$-ideals in $C(K)$ (= ring of all continuous real-valued functions on a compact Hausdorff space) are precisely either the essential ideals or the minimal ideals singly generated by the characteristic function of an isolated point.

? Some structural aspects?

? Some structural aspects?

- A ring with no proper non-zero $\mathscr{M}$-ideals is simple.

Some structural aspects?

- A ring with no proper non-zero $\mathscr{M}$-ideals is simple.

- If J is an $\mathscr{M}$-ideal of R, and K is an ideal of R which is contained in J, then J/K is an $\mathscr{M}$-ideal of R/K.

? Some structural aspects?

- A ring with no proper non-zero $\mathscr{M}$-ideals is simple.

- If J is an $\mathscr{M}$-ideal of R, and K is an ideal of R which is contained in J, then J/K is an $\mathscr{M}$-ideal of R/K.

- If I is an $\mathscr{M}$-ideal in a regular ring A and if $I \subseteq B \subseteq A$ with B a subring, then I is an $\mathscr{M}$-ideal in B.

? Some structural aspects?

- A ring with no proper non-zero $\mathscr{M}$-ideals is simple.

- If J is an $\mathscr{M}$-ideal of R, and K is an ideal of R which is contained in J, then J/K is an $\mathscr{M}$-ideal of R/K.

- If I is an $\mathscr{M}$-ideal in a regular ring A and if $I \subseteq B \subseteq A$ with B a subring, then I is an $\mathscr{M}$-ideal in B.

- Suppose that a unital ring R has a nontrivial ideal decomposition $R = I \oplus J$. Then I is an $\mathscr{M}$-ideal in R if and only if I is relatively irreducible.

? Some structural aspects?

- A ring with no proper non-zero $\mathscr{M}$-ideals is simple.
- If J is an $\mathscr{M}$-ideal of R, and K is an ideal of R which is contained in J, then J/K is an $\mathscr{M}$-ideal of R/K.
- If I is an $\mathscr{M}$-ideal in a regular ring A and if $I \subseteq B \subseteq A$ with B a subring, then I is an $\mathscr{M}$-ideal in B.
- Suppose that a unital ring R has a nontrivial ideal decomposition $R = I \oplus J$. Then I is an $\mathscr{M}$-ideal in R if and only if I is relatively irreducible.
- For any ring R, the following are equivalent:
  1. Every proper $\mathscr{M}$-ideal of R is a direct summand of R.
  2. Every proper $\mathscr{M}$-ideal of R is a simple ideal of R which is also a direct summand.
  3. Every proper ideal of R is a direct summand of R.
  4. R is a direct sum of simple rings.

? Cloning results from essential submodules?

? Cloning results from essential submodules?

If N and N' are ideals of R such that N is an $\mathscr{M}$-ideal of N' and N' is an $\mathscr{M}$-ideal of R, then N is an $\mathscr{M}$-ideal of R.

Cloning results from essential submodules?

If N and N' are ideals of R such that N is an $\mathscr{M}$-ideal of N' and N' is an $\mathscr{M}$-ideal of R, then N is an $\mathscr{M}$-ideal of R.

Suppose that R is a regular ring. Let A_1, A_2, B_1, B_2 be ideals of R. If A_1 is an $\mathscr{M}$-ideal of B_1 and A_2 is an $\mathscr{M}$-ideal of B_2, then $A_1 \cap A_2$ is an $\mathscr{M}$-ideal of $B_1 \cap B_2$.

? Cloning results from essential submodules?

If N and N' are ideals of R such that N is an $\mathscr{M}$-ideal of N' and N' is an $\mathscr{M}$-ideal of R, then N is an $\mathscr{M}$-ideal of R.

Suppose that R is a regular ring. Let A_1, A_2, B_1, B_2 be ideals of R. If A_1 is an $\mathscr{M}$-ideal of B_1 and A_2 is an $\mathscr{M}$-ideal of B_2, then $A_1 \cap A_2$ is an $\mathscr{M}$-ideal of $B_1 \cap B_2$.

An *$\mathscr{M}$-complement* of an ideal N of a ring R is an ideal N' of R such that $N \cap N' = 0$ and $N + N'$ is an $\mathscr{M}$-ideal of R.

? Cloning results from essential submodules?

If N and N' are ideals of R such that N is an $\mathscr{M}$-ideal of N' and N' is an $\mathscr{M}$-ideal of R, then N is an $\mathscr{M}$-ideal of R.

Suppose that R is a regular ring. Let A_1, A_2, B_1, B_2 be ideals of R. If A_1 is an $\mathscr{M}$-ideal of B_1 and A_2 is an $\mathscr{M}$-ideal of B_2, then $A_1 \cap A_2$ is an $\mathscr{M}$-ideal of $B_1 \cap B_2$.

An *$\mathscr{M}$-complement* of an ideal N of a ring R is an ideal N' of R such that $N \cap N' = 0$ and $N + N'$ is an $\mathscr{M}$-ideal of R.

$\mathscr{M}$-complements are not unique: In $\mathbb{Z}_{12}$, (3) and (6) are $\mathscr{M}$-complements of (4).

...Continued

...Continued

If A and N are ideals of a regular arithmetic ring R such that $A \subseteq N$, and B is an $\mathscr{M}$-complement of A in R, then $B \cap N$ is an $\mathscr{M}$-complement of A in N.

...Continued

If A and N are ideals of a regular arithmetic ring R such that $A \subseteq N$, and B is an $\mathscr{M}$-complement of A in R, then $B \cap N$ is an $\mathscr{M}$-complement of A in N.

If N and Q are ideals of a ring R with $N \cap Q = 0$, then N has an $\mathscr{M}$-complement containing Q.

...Continued

If A and N are ideals of a regular arithmetic ring R such that $A \subseteq N$, and B is an $\mathscr{M}$-complement of A in R, then $B \cap N$ is an $\mathscr{M}$-complement of A in N.

If N and Q are ideals of a ring R with $N \cap Q = 0$, then N has an $\mathscr{M}$-complement containing Q.

In a ring R, every ideal N of R has an $\mathscr{M}$-complement.

...Continued

If A and N are ideals of a regular arithmetic ring R such that $A \subseteq N$, and B is an $\mathscr{M}$-complement of A in R, then $B \cap N$ is an $\mathscr{M}$-complement of A in N.

If N and Q are ideals of a ring R with $N \cap Q = 0$, then N has an $\mathscr{M}$-complement containing Q.

In a ring R, every ideal N of R has an $\mathscr{M}$-complement.

If a ring R does not have any proper $\mathscr{M}$-ideal, then R is complemented.

Quantalic properties?

Quantalic properties?

Let $\varphi\colon L \to L'$ be an injective quantale homomorphism. If y is a $\mathscr{M}$-element in L', then $\varphi^{-1}(y)$ is a $\mathscr{M}$-element in L.

? Quantalic properties?

Let $\varphi\colon L \to L'$ be an injective quantale homomorphism. If y is a $\mathscr{M}$-element in L', then $\varphi^{-1}(y)$ is a $\mathscr{M}$-element in L.

Let L be a frame and $a \in L$. Consider the map $\kappa_a\colon L \to a^{\uparrow}$ defined by

$$\kappa_a(x) := x \vee a.$$

? Quantalic properties?

ⓘ Let $\varphi\colon L \to L'$ be an injective quantale homomorphism. If y is a $\mathscr{M}$-element in L', then $\varphi^{-1}(y)$ is a $\mathscr{M}$-element in L.

ⓘ Let L be a frame and $a \in L$. Consider the map $\kappa_a\colon L \to a^{\uparrow}$ defined by

$$\kappa_a(x) := x \vee a.$$

The following properties are equivalent:

1. κ_a preserves essential elements.
2. a is regular*.
3. κ_a preserves essential elements and $\mathscr{M}$-elements.

? Quantalic properties?

Let $\varphi\colon L \to L'$ be an injective quantale homomorphism. If y is a $\mathscr{M}$-element in L', then $\varphi^{-1}(y)$ is a $\mathscr{M}$-element in L.

Let L be a frame and $a \in L$. Consider the map $\kappa_a\colon L \to a^{\uparrow}$ defined by

$$\kappa_a(x) := x \vee a.$$

The following properties are equivalent:

1. κ_a preserves essential elements.
2. a is regular*.
3. κ_a preserves essential elements and $\mathscr{M}$-elements.

*An element $a \in L$ is said to be regular if $a^{\perp\perp} = a$, where $a^{\perp} := \bigvee\{x \in L \mid xa = 0\}$.

? What next?

? What next?

- Develop $\mathscr{M}$-submodule theory similar to injective hulls.

? What next?

- Develop $\mathscr{M}$-submodule theory similar to injective hulls.
- Characterize $\mathscr{M}$-ideals for:

? What next?

- Develop $\mathscr{M}$-submodule theory similar to injective hulls.
- Characterize $\mathscr{M}$-ideals for:
  - Rings of measurable functions.

? What next?

- Develop $\mathscr{M}$-submodule theory similar to injective hulls.
- Characterize $\mathscr{M}$-ideals for:
  - Rings of measurable functions.
  - Polynomial rings.

What next?

- Develop $\mathscr{M}$-submodule theory similar to injective hulls.
- Characterize $\mathscr{M}$-ideals for:
  - Rings of measurable functions.
  - Polynomial rings.
  - Topological rings.

? What next?

- Develop $\mathscr{M}$-submodule theory similar to injective hulls.
- Characterize $\mathscr{M}$-ideals for:
  - Rings of measurable functions.
  - Polynomial rings.
  - Topological rings.
  - ...

? What next?

- Develop $\mathscr{M}$-submodule theory similar to injective hulls.
- Characterize $\mathscr{M}$-ideals for:
  - Rings of measurable functions.
  - Polynomial rings.
  - Topological rings.
  - ...
- Introduce and study $\mathscr{M}$-ideals in other algebraic structures (semirings, monoids, associative algebras,...)


thank you