Lorenzo Guerrieri

Jagiellonian University in Krakow

Egyptian fractions and reciprocal complements of integral domains

Conference on Rings and Polynomials - Graz July 2025

1. Egyptian fractions in integral domains.

Classically, an Egyptian fraction is a representation of a rational number as a sum of distinct unit fractions.

Theorem.
Let q be a positive rational number and $N \geq 0$ an integer. Then there exist $n_1 > n_2 > \ldots > n_c > N$, such that

$$q = \frac{1}{n_1} + \ldots + \frac{1}{n_c}.$$

Definition (LG - Loper - Oman). Let D be an integral domain. An nonzero element $x \in D$ is *Egyptian* if there exist (distinct) nonzero elements $d_1, \ldots, d_n \in D$ such that

$$x = \frac{1}{d_1} + \ldots + \frac{1}{d_n}.$$

The ring D is an *Egyptian domain* if all its nonzero elements are Egyptian.

Remark. Thanks to the fact that integers can be represented as Egyptian fractions with arbitrarily large denominators, the assumption of distinct denominators is redundant for integral domains.

The ring of integers $\mathbb{Z}$ is an Egyptian domain.

Overrings and integral extensions of Egyptian domains are also Egyptian.

Let us denote by E the set of Egyptian elements of D. Observe that:

- If x is a unit in D, then $x = \frac{1}{x^{-1}} \in E$.
- If $x, y \in E$, then $x + y \in E \cup \{0\}$.
- If $x, y \in D$, then $x, y \in E$ if and only if $xy \in E$.

Integral domains with nonzero Jacobson radical are Egyptian. To prove it, pick $x \in D$, $y \in J(D) \setminus \{0\}$, and use that $xy = (xy + 1) - 1$ is a sum of units, hence Egyptian.

The polynomial ring $K[X]$ is not Egyptian. Indeed, if

$$X = \frac{1}{f_1} + \ldots + \frac{1}{f_n},$$

set $\delta_i = \deg(f_i) \geq 0$, and suppose $\delta_1 \leq \delta_2 \leq \ldots \leq \delta_n$. Set also $F = \prod_{i=1}^{n} f_i$.

Then

$$1 + \sum_{i=1}^{n} \delta_i = \deg(XF) = \deg \left(\sum_{i=1}^{n} \frac{F}{f_i} \right) \leq \sum_{i=2}^{n} \delta_i.$$

This is a contradiction.

Similarly, one gets that polynomial rings and semigroup algebras over arbitrary domains are not Egyptian (if the semigroup is not a group).

However, it is interesting to notice that every overring of $K[X]$ is Egyptian.

2. Reciprocal complements of integral domains

We know that $\mathbb{Z}$ and $K[X]$ are both Euclidean domains. But one of them is Egyptian and the other is not. Motivated by understanding this difference, Epstein gave this definition:

Definition (Epstein). Let D be an integral domain with quotient field $\mathcal{Q}(D)$. The *reciprocal complement* $R(D)$ is the subring of $\mathcal{Q}(D)$ generated by all fractions $\frac{1}{d}$ for nonzero $d \in D$.

Remark. D is Egyptian if and only if $R(D) = \mathcal{Q}(D)$.

We have

$$R(\mathbb{Z}) = \mathbb{Q}, \quad R(K[X]) = \left\{ \frac{f}{g} \mid f, g \in K[X], \, \deg(f) \leq \deg(g) \right\} = K[X^{-1}]_{(X^{-1})}.$$

Theorem (Epstein).
If D is an Euclidean domain, then $R(D)$ is either a field or a DVR.

We can obtain a sort of converse of the previous theorem.

Let S be a multiplicatively closed subset of D. Then

$$R(S^{-1}D) = R(D)[S].$$

Hence, $R(E^{-1}D) = R(D)[E] = R(D)$ and to study $R(D)$ we can always reduce to the case where the Egyptian elements of D are invertible (and together with zero form a field).

Theorem (LG).
Suppose that $E \cup \{0\} = K$ is a field. Then, $R(D)$ is a DVR if and only if $D \cong K[X]$.

3. Prime ideals of reciprocal complements

Let D be an integral domain. We have the following results:

- $R(D)$ is a local domain and its maximal ideal is generated by all $\frac{1}{x}$ for $x \in D \setminus (E \cup \{0\})$.

- For every nonzero $x \in D$, there exists a unique prime ideal $\mathfrak{p}_x$ of $R(D)$ maximal with respect to the property of excluding $\frac{1}{x}$. In particular, $R(D)_{\mathfrak{p}_x} = R(D)[x] = R(D[x^{-1}])$.

- If $\dim(R(D)) < \infty$, then for every prime $\mathfrak{p} \in \mathrm{Spec}(R(D))$, there exists $x \in D$ such that $\mathfrak{p} = \mathfrak{p}_x$.

- If $\dim(R(D)) < \infty$, there exists $x \in D$ such that $(0) = \mathfrak{p}_x$ and therefore $\frac{1}{x}$ is contained in every nonzero prime ideal of $R(D)$.

- If $\dim(R(D)) \geq 2$, then $R(D)$ is not Noetherian.

Conjecture. $\dim(R(D)) \leq \dim(D)$.

This conjecture is proved for:

- finitely generated algebras over a field,
- semigroup algebras of the form $K[S]$ with S contained in the positive part of a totally ordered abelian group.

For any given $n \geq m \geq 0$, it is possible to find D such that $\dim(D) = n$ and $\dim(R(D)) = m$ (and also $E \cup \{0\}$ is a field).

If $D = K[X_1, X_2, \ldots,]$, then $\dim(R(D)) = \infty$ and $(0) \neq \mathfrak{p}_x$ for any $x \in D$.

4. Reciprocal complements of polynomial rings in several variables.

In joint work with Epstein and Loper, we studied $R_n := R(K[X_1, \dots, X_n])$.

This ring has a "polynomial-like behavior":

- For $i < n$, $R_n \cap K(X_1, \dots, X_i) = R_i$.
- For $i < n$, there exists prime ideals $\mathfrak{q}_i$ such that

$$\frac{R_n}{\mathfrak{q}_i} \cong R_i, \quad (R_n)_{\mathfrak{q}_i} = R(K(X_1, \dots, X_i)[X_{i+1}, \dots, X_n]).$$

- $\dim(R_n) = n$.
- R_n has infinitely many prime ideals of every height $i = 1, \dots, n-1$.

But also a very different behavior:

- The element $\frac{1}{X_1 X_2 \cdots X_n} \in \mathfrak{p}$ for every nonzero prime $\mathfrak{p}$ of R_n.
- For $n \geq 2$, R_n is not Noetherian.
- For $n \geq 2$, R_n is not integrally closed.

Observe that for $a, b \in D$ (with $b, a+b \neq 0$) we have

$$\frac{a}{b} \cdot \frac{1}{a+b} = \frac{1}{b} - \frac{1}{a+b} \in R(D).$$

If $\beta = \frac{X}{Y} \cdot \frac{1}{X^3+Y^2}$, then

$$\beta^2 = \frac{1}{X} \cdot \frac{X^3}{Y^2} \cdot \frac{1}{X^3+Y^2} \cdot \frac{1}{X^3+Y^2} \in R_2.$$

However $\beta \notin R_2$. To show this we constructed a valuation overring $V \supseteq R_2$ such that $v(\beta) \notin v(R_2)$.

Further open questions. Let $n \geq 2$.

- Describe the elements of the integral closure of R_n, establish whether is local, completely integrally closed, etc..

- Find an algorithm to determine whether a given rational function $\frac{f}{g} \in R_n$ for $f, g \in K[X_1, \ldots, X_n]$.

5. Factorization.

Also the factorization properties of R_n (for $n \geq 2$) are not yet known.

We know that:

Theorem. The ring R_n is atomic.

Proof. R_n is dominated by the DVR

$$\left\{ \frac{f}{g} \mid f, g \in K[X_1, \ldots, X_n], \, \mathrm{deg}(f) \leq \mathrm{deg}(g) \right\}.$$

Factorization in R_n is not unique:

$$\frac{1}{X} \cdot \frac{1}{Y} = \frac{1}{X+Y} \cdot \left(\frac{1}{X} + \frac{1}{Y} \right).$$

6. Comparison of $R(K[X,Y])$ and $K[X,Y]_{(X,Y)}$.

These two rings are both local and two-dimensional.

For $K[X,Y]_{(X,Y)}$, it is well-known that:

- Every nonzero (finitely generated) ideal is contained in only finitely many primes.
- Localizations at height one primes are DVRs.
- Quotients at height one primes are Noetherian, but often not regular.

For $R(K[X,Y])$, we can prove that:

- Every finitely generated ideal is contained in all but finitely many primes.
- Localizations at height one primes are Noetherian, but often not regular.
- Quotients at height one primes are DVRs (still unpublished).

Bibliography:

N. Epstein, *The unit fractions from a Euclidean domain generate a DVR,* Ric. Mat. (2024). https://doi.org/10.1007/s11587-024-00922-0

N. Epstein, LG, K. A. Loper, *The reciprocal complement of a polynomial ring in several variables over a field,* To appear on Pacific Journal of Mathematics. ArXiv:2407.15637 (2024)

LG, *The reciprocal complements of classes of integral domains,* Journal of Algebra 682 (2025) 188–214.

LG, K. A. Loper, G. Oman, *From ancient Egyptian fractions to modern algebra,* Journal of Algebra and its Applications (2025) https://doi.org/10.1142/ S0219498826500787

Thanks for your attention!