Polynomial Rings and Coordinates

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Rings and Polynomials
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Challenging Problems on Polynomial Rings

Polynomial rings present several challenging open problems — easy to state (at least for mathematicians), difficult to solve.

1. **Jacobian Conjecture** (O.H. Keller).
2. **Epimorphism Problem** (Abhyankar-Sathaye).
3. $\mathbb{A}^n$-**Fibration Problem** (Dolgachev-Weisfeiler).
4. **Zariski Cancellation Problem**.
5. **Linearisation Problem** (Kambayashi).
6. **Characterisation Problem**.
7. $\mathbb{A}^n$-**form Problem**.

Notation and Main Themes

Throughout my talk, k will denote a field of any characteristic with algebraic closure $\bar{k}$.

For a ring R,
$A = R^{[n]}$ denotes $A = R[F_1, \ldots, F_n]$ for some $F_1, \ldots, F_n \in A$ which are algebraically independent over R i.e., A is a polynomial ring in n indeterminates over R.

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• To determine whether a polynomial F is a *coordinate* of $k[X_1, \ldots, X_n]$, i.e., whether there exist $F_2, \ldots, F_n$ such that $k[X_1, \ldots, X_n] = k[F, F_2, \ldots, F_n] = k[F]^{[n-1]}$.

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Of special interest: rings A which are quotients of polynomial rings by linear polynomials.

A Pioneer in Affine Algebraic Geometry

Shreeram S. Abhyankar (1930-2012)

Polynomials and power series,
May they forever rule the world

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k: field of characteristic 0.

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$T = g(u(T), v(T))$ for some polynomial $g(X, Y)$.

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Thm 2 (Abhyankar-Moh, Suzuki). Let $F \in k[X, Y]$. Then

$$k[X, Y]/(F) = k^{[1]} \Rightarrow k[X, Y] = k[F]^{[1]}.$$

Epimorphism Problem

Q. $$\frac{\mathbf{k}[\mathbf{X_1}, \ldots, \mathbf{X_n}]}{(\mathbf{F})} = \mathbf{k}^{[\mathbf{n}-1]} \implies \mathbf{k}[\mathbf{X_1}, \ldots, \mathbf{X_n}] = \mathbf{k}[\mathbf{F}]^{[\mathbf{n}-1]}?$$

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$n = 2$: **YES** ch $k = 0$ (Abhyankar-Moh; Suzuki (1975))

$n = 2$: **NO** ch $k > 0$ (Segre 1957, Nagata (1972))

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Let $g(Z, T) = Z^{p^e} + T + T^{sp} \in k[Z, T]$,

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Question on Epimorphism Problem can be asked even when ch $k \geq 0$ and F is of certain specified type.

Theorem on Linear Planes

Thm (Sathaye, Russell (1976)): Let ch $k \geq 0$ and

$$\frac{k[X, Y, Z]}{(G)} = k^{[2]}, \quad \text{where} \quad G = a(X, Z)Y - b(X, Z).$$

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Then $k[X, Y, Z] = k[G]^{[2]}$ and there exists $X_1 \in k[X, Z]$ s.t.

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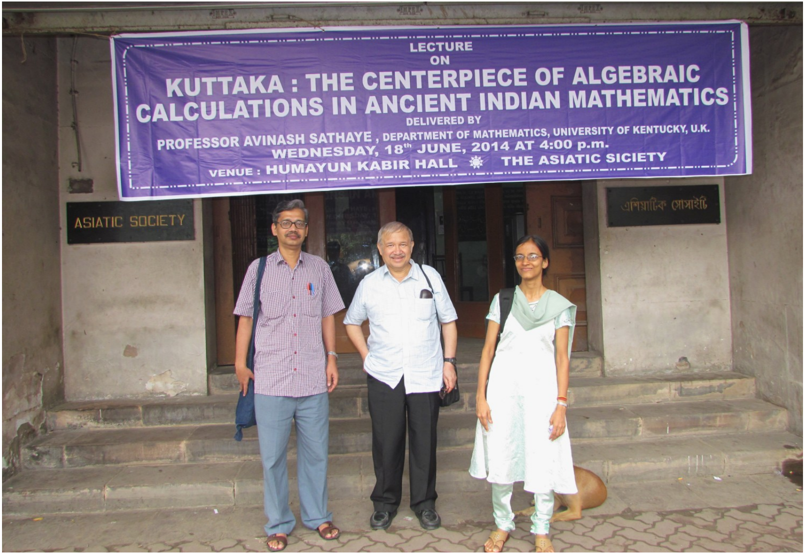
Q. Let $\dfrac{k[X,Y,Z,T]}{(G)} = k^{[3]}$ where $G = a(X)Y - b(X,Z,T)$.

Is $k[X,Y,Z,T] = k[X,G]^{[2]}$?

YES for $a(X) = X^r$ where $r > 1$ (— (2014)).



A.K. Dutta, A. Sathaye and N. Gupta
at the Asiatic Society, Kolkata

A few other cases

Thm: k field of characteristic $p \geq 0$ and $F = aY^n - b$, where $a, b \in k[X, Z]$ and $p \nmid n$. Then

$$\mathbf{k[X, Y, Z]/(F) = k^{[2]} \implies k[X, Y, Z] = k[F]^{[2]}},$$

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Thm (Kaliman (2002)): Suppose that $G \in \mathbb{C}[X, Y, Z]$ s.t.

$$\frac{\mathbb{C}[\mathbf{X, Y, Z}]}{(\mathbf{G} - \lambda)} = \mathbb{C}^{[2]} \quad \text{for almost all} \quad \lambda \in \mathbb{C}.$$

Then $\mathbb{C}[\mathbf{X, Y, Z}] = \mathbb{C}[\mathbf{G}]^{[2]}$.

Generalisation of Sathaye-Russell Linear Planes

k: any field,
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For $m \geq 1$ and general k, affirmative answers under certain assumptions on α and F are given in recent papers with Parnashree Ghosh and Ananya Pal.
Example: (i) Ch $k = 0$ and $F \in k[Z, T]$.
(ii) Ch $k \geq 0$, $\alpha = a_1(X_1)a_2(X_2)\cdots a_m(X_m)$ and $F \in k[Z, T]$.

Parnashree Ghosh at the 80th birthday conference in the honour of H. Kraft, Monte-Verita

Ananya Pal at St. Petersburg, Russia

Thm (Ghosh, —, Pal)

k *any* field, $\mathbf{F = a(X)Y - b(X, Z, T) \in k[X, Y, Z, W]}$, $\mathbf{a \neq 0}$,

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and x: image of X in A. Suppose $a(X)$ has no simple root in $\overline{k}$. Then the following statements are equivalent:

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k *any* field, $\mathbf{F} = \mathbf{a}(\mathbf{X})\mathbf{Y} - \mathbf{b}(\mathbf{X}, \mathbf{Z}, \mathbf{T}) \in \mathbf{k}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}]$, $\mathbf{a} \neq \mathbf{0}$,

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Eg: The following polynomials do not define affine 3-spaces

- $G_1 = X^2(X+1)^2Y - (Z^2 + T^3) - X \in k_1[X, Y, Z, T]$.
- $G_2 = (X^p - \lambda^p)Y - (Z^2 + T^3) + X \in k_2[X, Y, Z, T]$,

where ch. $k_2 = p > 0$, $\lambda^p \in k_2 \setminus k_2^p$.

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$f \neq 0$ and every prime divisor of α divides h and

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Suppose $A^{[l]} = k^{[l+m+2]}$ for some $l \geqslant 0$
and that $k[Z, T]/(f)$ is a regular ring. Then

$$k[Z, T] = k[f]^{[1]}$$

and

$$B = k[X_1, \ldots, X_m, H]^{[2]}.$$

Why linear polynomials?

Breakthroughs on central problems in AAG involved varieties defined by "linear" polynomials of the form $F = aY - b$, where $a \in k[X]$ and $b \in k[X, Z, T]$.

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• Solution of Linearization Problem for $\mathbb{C}^*$-actions on $\mathbb{C}^3$ involved questions like:

Q. Is the Russell-cubic $A = \dfrac{\mathbb{C}[X, Y, Z, T]}{(X^2Y + X + Z^2 + T^3)} = \mathbb{C}^{[3]}$?

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Thm (Makar-Limanov (1996)): NO .

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Q. Is the Russell-cubic $A = \dfrac{\mathbb{C}[X, Y, Z, T]}{(X^2Y + X + Z^2 + T^3)} = \mathbb{C}^{[3]}$?

Thm (Makar-Limanov (1996)): NO .

• I could give negative solution to ZCP in +ve ch. for 3-space by proving that the "Asanuma threefold"

$$\mathbf{A} = \frac{\mathbf{k[X, Y, Z, T]}}{(\mathbf{X^rY + Z^{p^2} + T + T^{sp}})} = \mathbf{k}^{[3]}, \text{ ch. } k = p, \; p \nmid s \,; r, s > 1.$$

The threefold was earlier involved in questions on the Affine Fibration Problem and the Linearisation Problem in +ve ch.

Affine Fibration Problem

Let R be a ring, P a prime ideal of R and A an R-algebra.
Notation:
$k(P)$: the field of fractions of the integral domain R/P.
$A \otimes_R k(P)$: the fibre ring of A at P.

Aim: To extract information about the R-algebra A from data on its fibre rings $A \otimes_R k(P)$.

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Question (Dolgachev and Weisfeiler (1974)):
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Affine Fibration Problem: Some Theorems

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Thm (**Asanuma (1987)**): Let R be a regular local ring and A an $\mathbb{A}^n$-fib over R. Then $A^{[\ell]} = R^{[n+\ell]}$ for some ℓ.

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Ex: Let k be a field of characteristic $p > 0$ and

$$\mathbf{A} = \frac{\mathbf{k[X, Y, Z, T]}}{(\mathbf{X^r Y + Z^{p^2} + T + T^{sp}})}, \quad p \nmid s \quad , s \geq 2, r \geq 1.$$

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P. Russell called this dichotomy: **Asanuma's Dilemma**.

A stalwart in Affine Algebraic Geometry

T. Asanuma at Ramakrishna Mission Institute of Culture, Kolkata

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Thm (— 2014): $\mathbf{A \ncong k^{[3]}}$ for $r \geq 2$.

A Founder of modern Algebraic Geometry

Oscar Zariski (1899-1986)
Brought rigour in classical algebraic geometry, laid the foundation of modern algebraic Geometry with A. Weil, connected it with commutative algebra

Zariski Cancellation Problem

Zariski Cancellation Problem: Is $\mathbb{A}_k^n$ cancellative as an affine variety? i.e., for an affine variety $\mathbb{V}$,

$$\mathbb{V} \times \mathbb{A}_{\mathbf{k}}^{\mathbf{1}} \cong \mathbb{A}_{\mathbf{k}}^{\mathbf{n+1}} \implies \mathbb{V} \cong \mathbb{A}_{\mathbf{k}}^{\mathbf{n}}?$$

More generally, is $k^{[n]}(= k[X_1, \ldots, X_n])$ cancellative? i.e.,

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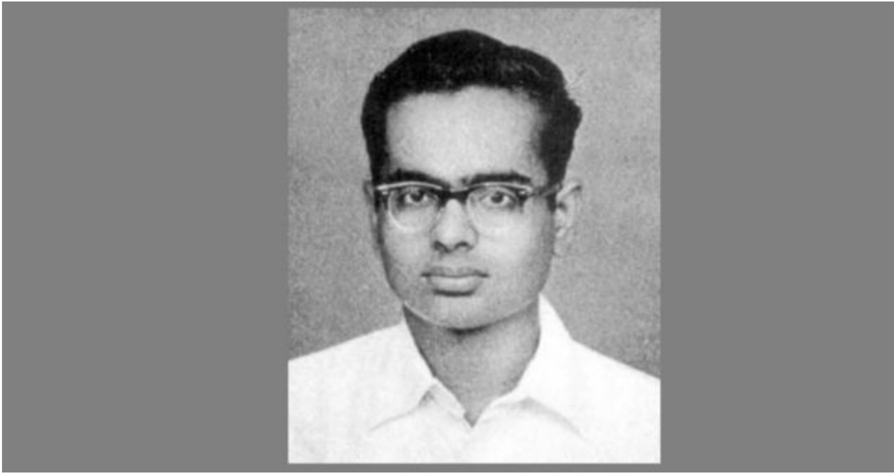
Research on ZCP for $n = 2$ had led to:

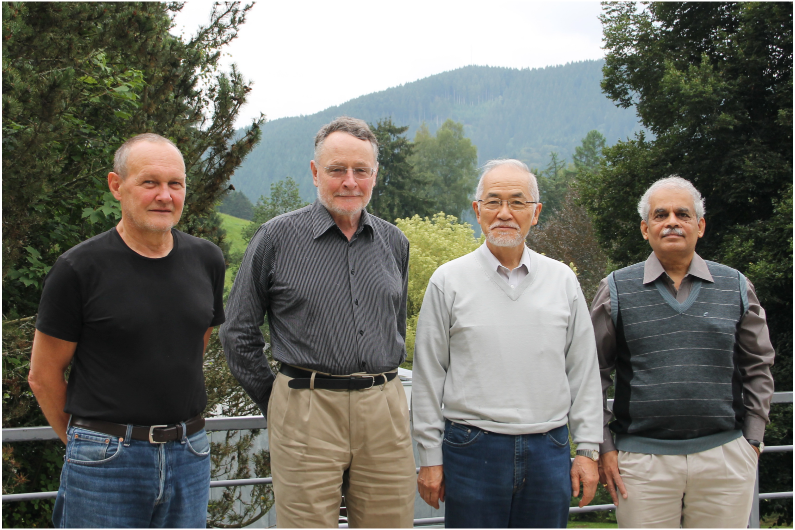
- Topological characterisation of $\mathbb{C}^2$ (C.P. Ramanujam (1971))
- Algebraic characterisation of k^2 (M. Miyanishi (1975))

A pioneer on the Characterisation Problem



C.P. Ramanujam (1938-1974)

Four Stalwarts of Affine Algebraic Geometry



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Research on ZCP opened up its connection with important problems and concepts in Affine Algebraic Geometry like Embedding Problem and Affine Fibration Problem.

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An *Exponential map* on a ring B is a ring homomorphism

$$\phi_U : B \to B[U] \text{ satisfying}$$

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Corollary. $A \subseteq B$. Then TFAE:
(I) $B = A^{[1]}$.
(II) $\exists \ D \in \mathrm{LND}(B)$ s.t. $1 \in D(B)$.

LND and Exponential Map

Let $\mathbb{Q} \subseteq B$. Any exponential map $\phi_U : B \to B[U]$ mapping

$$b \longmapsto b + b_1 U + b_2 U^2 + \dots$$

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Ring of invariants $\Longleftrightarrow$ Ker of D.

Some invariants

B: k-algebra
$\mathrm{Exp}_k(B)$: set of all k-linear exponential maps on B
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If $\mathbb{Q} \subseteq k \subseteq B$, then

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Characterisation Problem: dim ≤ 2

Some characterisations of $\mathbb{A}^1_{\mathbb{C}}$

- The only smooth contractible affine curve is $\mathbb{A}^1_{\mathbb{C}}$.
- The only one-dim affine UFD with trivial units is $\mathbb{C}^{[1]}$.

Topological characterisation of $\mathbb{A}^2_{\mathbb{C}}$ (C. P. Ramanujam (1971)):

- The only smooth contractible affine surface which is simply connected at infinity is $\mathbb{A}^2_{\mathbb{C}}$.
  In particular, any smooth contractible affine surface homeo to $\mathbb{R}^4$ is $\mathbb{A}^2_{\mathbb{C}}$.

Algebraic characterisation of $\mathbb{A}^2_{\mathbb{C}}$ (Miyanishi (1975)):

- The only two-dim factorial affine $\mathbb{C}$-domain A with trivial units s.t. $\mathrm{Spec}(A)$ contains a cylinder-like open set is $\mathbb{C}^{[2]}$.

Application (Fujita-Miyanishi-Sugie): $\mathbb{A}^2_{\mathbb{C}}$ **is cancellative**.
Gurjar (2002) gave a proof extending Ramanujam's ideas.

Algebraic characterisation of $\mathbb{A}_k^2$ (Dasgupta — (2019))
Let B be an affine k-domain of dim 2. TFAE:

(i) $B = k^{[2]}$.
(ii) $ML^*(B) = k$.
(iii) $ML(B) = k$ and $ML^*(B) \neq B$.

Remark: Thm does not hold when dim $B = 3$.

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k alg closed field and B an affine UFD of dim 3. TFAE:

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Remark: Thm does not hold when dim $B = 4$.

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$\mathbb{A}^n$-forms

k a field of characteristic $p \geq 0$ with algebraic closure $\bar{k}$.
A k-algebra A is called an *$\mathbb{A}^n$-form over k* if

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Example of a non-trivial $\mathbb{A}^1$-form:

Suppose $k \neq k^p$ and $\beta \in k \setminus k^p$.
Let $\alpha \in \bar{k}$ be such that $\beta = \alpha^p$, and

$$A = \frac{k[X, Y]}{(Y^p - X - \beta X^p)}.$$

Then, $A \otimes_k \bar{k} = \bar{k}^{[1]}$ but $A \neq k^{[1]}$.

Affine forms over fields of char. zero

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- $n = 1$: **YES** (Classical).
- $n = 2$: **YES** (T. Kambayasi (1975)).
- $n \geq 3$: OPEN

Partial Affirmative Answers when $n = 3$ and:

(i) A admits a fixed point free locally nilpotent derivations (Daigle–Kaliman (2009)).

(ii) A contains an element f which is a coordinate in $A \otimes_k \bar{k}$ (Daigle–Kaliman (2009)).

(iii) A admits an effective action of a reductive algebraic k-group of positive dimension (Koras-Russell (2013)).

(iv) A admits a non-confluent action of a unipotent group of dimension two (Gurjar–Masuda–Miyanishi).

Theorem (Dutta- — -Lahiri (2020)): $\mathbb{A}^3$-forms

k: field of characteristic zero
$\bar{k}$: algebraic closure of k
A: an affine k-domain
Suppose

$$A \otimes_k \bar{k} = \bar{k}^{[3]},$$

and there exists a locally nilpotent derivation D on A satisfying $\mathrm{rk}\,(D \otimes 1_{\bar{k}}) \leq 2$. Then

$$A = k^{[3]}.$$

r-divisible

For $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{Z}^m_{>0}$, $\alpha (\neq 0) \in k^{[m]}$ is called **r**-divisible if . . . .

For $m = 1$, $\alpha \in k[X_1]$ is $(\mathbf{r_1})$-divisible if

$$\alpha = X_1^{\mathbf{r_1}} \alpha_1(X_1) = X_1^{\mathbf{r_1}}(X_1 \beta_1 + \alpha_1(0)), \quad \alpha_2 := \alpha_1(0)) \in k^*$$

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$\alpha = X_1X_2^2(X_1 + X_2^2)^2$ is $(\mathbf{2}, \mathbf{3})$-divisible in X_2, X_1 and is $(\mathbf{1}, \mathbf{6})$-divisible in X_1, X_2.

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Prop (Ghosh — Pal)

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$B := k[X_1, \ldots, X_m, Y, Z, T]$,
$\mathbf{r} := (r_1, \ldots, r_m) \in \mathbb{Z}^m_{>1}$,
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Suppose either $\mathrm{ML}(A) = k$ or $\mathrm{DK}(A) = A$. Then there exist Z_1, T_1 of $k[Z, T]$ and $a_0, a_1 \in k^{[1]}$ such that

$$k[Z, T] = k[Z_1, T_1]$$

and

$$f(Z, T) = a_0(Z_1) + a_1(Z_1)T_1.$$

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Let $x_1, \ldots, x_m$ be the images of $X_1, \ldots, X_m$ in A respectively and $E := k[x_1, \ldots, x_m]$.
Then the following statements are equivalent:

- $k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, H]^{[2]}$.
- $k[X_1, \ldots, X_m, Y, Z, T] = k[H]^{[m+2]}$.
- $A = k[x_1, \ldots, x_m]^{[2]}$.
- $A = k^{[m+2]}$.
- $k[Z, T] = k[f(Z, T)]^{[1]}$.
- $A^{[l]} = k^{[l+m+2]}$ for some $l \geqslant 0$ and $\mathrm{ML}(A) = k$.
- $f(Z, T)$ is a line in $k[Z, T]$ and $\mathrm{ML}(A) = k$.
- A is an $\mathbb{A}^2$-fibration over E and $\mathrm{ML}(A) = k$.
- $A \otimes_k \bar{k}$ is a UFD, $\mathrm{ML}(A) = k$ and $\left(\frac{k_1[Z,T]}{(f(Z,T))}\right)^* = \bar{k}^*$.
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The family of hypersurfaces given by

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for $r_i \geqslant 2$, $1 \leqslant i \leqslant m$ and

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This gives a unified treatment of several apparently different-looking questions which have been of long interest to mathematicians (including Cancellation, Epimorphism and Fibration problems).

Consequences of the Theorem

k: a field of ANY characteristic, $\mathbf{r_i > 1}$, for $1 \leq i \leq m$, $m \geq 1$.

$$A := K[X_1, \ldots, X_m, Y, Z, T]/(aY - F),$$

where $a = {\pi_1}^{s_1} \ldots {\pi_n}^{s_n} \in k[X_1, \ldots, X_m]$ is **r**-divisible where $\mathbf{r} := (r_1, \ldots, r_m)$, π_i's primes,
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Proves a partial case of the Abhyankar-Sathaye Conjecture.
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- $A = k^{[m+2]}$ if and only if $A = k[x_1, \ldots, x_m]^{[2]}$.
- A is a non-trivial $\mathbb{A}^2$-fibration over $k[x_1, \ldots, x_m]$ if and only if $f(Z, T)$ is a non-trivial line.

Thm (— 2014)

Let R be a ring, $\pi_1, \pi_2, \ldots, \pi_n \in R$, $\pi := \pi_1 \pi_2 \cdots \pi_n$ and $G(Z, T) \in R[Z, T]$ be such that

$$R[Z, T]/(\pi, G(Z, T)) \cong (R/\pi)^{[1]}.$$

Let

$$D := R[Z, T, Y]/({\pi_1}^{s_1} {\pi_2}^{s_2} \cdots {\pi_n}^{s_n} Y - G(Z, T))$$

for any set of positive integers $s_1, \ldots, s_n$. Then

$$D^{[1]} = R^{[3]}.$$

Thm (Ghosh – Pal)

Let k be a field,

$$a = {\pi_1}^{s_1} \dots {\pi_n}^{s_n} \in k[X_1, \dots, X_m]$$

be an $\mathbf{r}$-divisible polynomial where $\mathbf{r} := (r_1, \dots, r_m)$, $\mathbf{r_i > 1}$ and

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where $f(Z, T)$ is a line. Let

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Thus, if $f(Z, T)$ is a non-trivial line, then A gives rise to a counter-example to the Zariski Cancellation Problem.

Theorem (Ghosh—, 2023)

k: a field of positive characteristic,

$$\mathbf{A(r_1, \ldots, r_m, f)} := \frac{\mathbf{k[X_1, X_2, \ldots, X_m, Y, Z, T]}}{\mathbf{(X_1^{r_1} \cdots X_m^{r_m} Y - f(Z, T))}},$$

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- $A(r_1, \ldots, r_m, f) \cong A(s_1, \ldots, s_m, g)$ iff $(r_1, \ldots, r_m)$ is equal to $(s_1, \ldots, s_m)$ up to permutation and f and g are equivalent.

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Thus, over a field k of positive characteristic, there is an infinite family of non-isomorphic rings which are stably isomorphic to $k^{[m+2]}$.

Generalised Epimorphism Theorems over Rings

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Thm (Bhatwadekar (1988)): Let R be a Noetherian ring of characteristic zero and $F \in R[X, Y]$. Then

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Ex: (Asanuma–Dutta (2021)): The hypotheses are necessary.

Theorem on Linear Planes over DVR

Thm (Bhatwadekar-Dutta (1994)):
Let (R, t) be a discrete valuation ring with field of fractions $K := R[1/t]$ and residue field $k := R/tR$. If $G = aZ - b \in R[X, Y][Z]$ is s.t.

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then there exists $X_0 \in R[X, Y]$ such that

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Is $R[X, Y, Z] = R[F]^{[2]}$? (OPEN)

Other Generalisations over Rings

Thm (Das-Dutta (2011)): Let R be a Noetherian domain and $F = aZ^n - b$, where $a, b \in R[X, Y]$. Then

$$\mathbf{R[X, Y, Z]/(F) = R^{[2]} \implies R[X, Y, Z] = R[F]^{[2]}},$$

whenever R contains $\mathbb{Q}$ and several other cases.

Thm (—2014): Let R be a Noetherian seminormal domain containing $\mathbb{Q}$ and $F = X^r Y - F(X, Z, T) \in R[X, Y, Z, T]$ for $r \geq 2$. Then

$$\mathbf{R[X, Y, Z, T]/(F) = R^{[3]} \implies R[X, Y, Z, T] = R[F]^{[3]}}.$$

Generalizations to higher dimensions by (Dutta, —(2015)), (Ghosh, —(2023)) and (Pal (2025)).

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If **NO**, then it is a counter-example to the following problems:

- $\mathbb{A}^2$-fibration Problem over $\mathbb{C}^{[2]}$;
- Cancellation Problem over $\mathbb{C}^{[1]}$;
- Epimorphism Problem for $\mathbb{C}^{[4]} \twoheadrightarrow \mathbb{C}^{[3]}$, $R^{[3]} \twoheadrightarrow R^{[2]}$.

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