Injectivity relative to closed submodules over Krull domains

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Overview

1. Injectivity relative to closed submodules

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3. w-Injectivity relative to closed submodules

Injectivity relative to closed submodules

Definition

Assume that R is a commutative ring with identity, and M is a unitary R-module. A submodule K of M is called closed in M if K has no proper essential extension in M.

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- X is called c-injective if X is M-c-injective for every R-module M.
- X is called self-c-injective if X is X-c-injective.

Some facts related to c-injective modules

Theorem (C. Santa-Clara and P.F. Smith, 2004)

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Theorem (E. Mermut, C. Santa-Clara, and P.F. Smith, 2009)

Let D be a Noetherian domain and P a maximal ideal of D. Then the following statements are equivalent.

1. The module D/P is c-injective.
2. The module D/P is M-c-injective, where $M = D \oplus D$.
3. P is an invertible ideal.

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Corollary (E. Mermut, C. Santa-Clara, and P.F. Smith, 2009)

A Noetherian integral domain D is Dedekind iff every simple D-module is c-injective.

Injectivity relative to closed submodules over Krull domains

Preliminaries

Assume that R is a commutative ring with the total quotient ring T.

Definition

A nonzero finitely generated ideal J of R is called a Glaz–Vasconcelos ideal, denoted by $J \in GV(R)$, if the natural homomorphism $\phi : R \to \mathrm{Hom}_R(J, R)$ is an isomorphism.

Assume that M is an R-module.

Definition

The complete GV-torsion submodule of M is defined as

$$\mathrm{tor}_{GV(R)}(M) = \{x \in M \mid Jx = 0 \text{ for some } J \in GV(R)\}.$$

If $\mathrm{tor}_{GV(R)}(M) = M$, then M is called a GV-torsion module.
If $\mathrm{tor}_{GV(R)}(M) = 0$, then M is called a GV-torsion-free module.

The w-envelope of modules

Assume that M is a GV-torsion-free R-module.

Definition

The *w-envelope* of M is the set given by

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in GV(R)\},$$

where $E(M)$ is the injective hull of M.

Definition

A GV-torsion-free module M is called a *w-module* if

$$\operatorname{Ext}^1_R(R/J, M) = 0$$

for any $J \in GV(R)$. Equivalently, M is w-module iff $M_w = M$.

Strong Mori domains

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Theorem (F. Wang and R.L. McCasland, 1999)

An integral domain D is strong Mori iff D_P is Noetherian for every maximal w-ideal P of D and each nonzero element of D lies in only finitely many maximal w-ideals.

Krull domains

Definition

An integral domain D is called a Krull domain if it satisfies the following three conditions:

(1) For every prime ideal P of D of height one, D_P is a discrete valuation ring.

(2) $D = \bigcap D_P$, where P ranges over all prime ideals of D of height one.

(3) Any nonzero element of D lies in only a finite number of prime ideals of height one.

Theorem (F. Wang and R.L. McCasland, 1999)

The following statements are equivalent for an integral domain D.

1. D is a Krull domain.
2. D is an integrally closed strong Mori domain.
3. D is a strong Mori domain and D_P is a DVR for each maximal w-ideal P of D.
4. D is a strong Mori domain and every maximal w-ideal of D is w-invertible.
5. Each nonzero ideal of D is w-invertible.

Main Results

Theorem

Assume that D is a strong Mori domain and P is a maximal w-ideal of D. Then the following statements are equivalent.

1. The module $(D/P)_w$ is c-injective.
2. The module $(D/P)_w$ is M-c-injective, where $M = D \oplus D$.
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Corollary

- Assume that D is a strong Mori domain. Then D is a Krull domain iff every w-simple D-module is c-injective.

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Corollary

- Assume that D is a strong Mori domain. Then D is a Krull domain iff every w-simple D-module is c-injective.
- Assume that D is a Krull domain. Then every direct product of torsion-free w-simple w-modules is self-c-injective.

w-Injectivity relative to closed submodules

Definition

Assume that R is a commutative ring. A sequence

$$A \to B \to C$$

of R-modules and R-homomorphisms is called w-exact if the sequence

$$A_P \to B_P \to C_P$$

is exact for any maximal w-ideal P of R.

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is exact for any maximal w-ideal P of R.

▷ A sequence $B \xrightarrow{f} C \to 0$ is w-exact iff $\mathrm{Coker}(f)$ is GV-torsion.

Definition

Assume that M is an R-module. Set

$$L(M) := (M/\mathrm{tor}_{GV}(M))_w.$$

An R-module M is said to be w-injective if

$$0 \to \mathrm{Hom}_R(C, L(M)) \to \mathrm{Hom}_R(B, L(M)) \to \mathrm{Hom}_R(A, L(M)) \to 0$$

is w-exact for any w-exact sequence

$$0 \to A \to B \to C \to 0.$$

Theorem (H. Kim and F. Wang, 2014)

A w-module M is w-injective if

$$0 \to \mathrm{Hom}_R(C, M) \to \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(A, M) \to 0$$

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Corollary (H. Kim and F. Wang, 2014)

Any GV-torsion-free injective module is a w-injective w-module.

c-w-Injectivity

Definition

We define a w-module E to be c-w-injective if for every closed submodule K of every torsion-free R-module M,

$$0 \to \mathrm{Hom}_R(M/K, E) \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(K, E) \to 0$$

is w-exact.

Clearly, any c-injective torsion-free w-module is c-w-injective.

Main results

Theorem

Let D be an integral domain and X a w-module. Then the following statements are equivalent.

1. For every closed ideal I of D, the sequence

$$0 \to \mathrm{Hom}_D(D/I, X) \to \mathrm{Hom}_D(D, X) \to \mathrm{Hom}_D(I, X) \to 0$$

   is exact.
2. X is c-injective.
3. X is c-w-injective.

Main results

Theorem

Let D be an integral domain and X a w-module. Then the following statements are equivalent.

1. For every closed ideal I of D, the sequence

$$0 \to \mathrm{Hom}_D(D/I, X) \to \mathrm{Hom}_D(D, X) \to \mathrm{Hom}_D(I, X) \to 0$$

   is exact.
2. X is c-injective.
3. X is c-w-injective.

Theorem

Let D be an integral domain and X a GV-torsion-free D-module. If X is a c-w-injective w-module, then $cX = X$ for any $c \in D$ with cD closed in D.

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Let D be an integral domain and X a GV-torsion-free D-module. If X is a c-w-injective w-module, then $cX = X$ for any $c \in D$ with cD closed in D.

Theorem

Let D be a Krull domain. If X is a GV-torsion-free w-module such that $cX = X$ for any $c \in D$ with cD closed in D, then X is c-injective.

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Thank you for your attention!