The monoid of relatively big projective modules

Dolors Herbera

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UAB
Universitat Autònoma
de Barcelona

Authorship and notation

The ideas in talk is based in different works that had been going on for quite some time, and what I want to present is part of the PhD Thesis of Román Álvarez that I am co-directing together with P. Príhoda (Charles University, Prague).

The formalization of the monoid of relatively big projectives was essentially done in a paper with P. Príhoda and R. Wiegand.

A starting point is a paper by P. Príhoda published in 2010.

In general we will work with, non-necessarily commutative algebras Λ, over a commutative ring (domain) R.

Local ring means a, non necessarily commutative ring, such that the set of non-units form an ideal.

Warning

H. Bass *Big projective modules are free*. Illinois J. Math. 7(1), 24-31, (1963)

Showed that non-finitely generated projective modules over an indecomposable commutative noetherian ring are always free.

Made a remark claiming that the theory of non-finitely generated projective modules *invites little interest*...

Want to explain that Bass' results are just showing part of the picture, and that already for finitely generated algebras over commutative noetherian rings there is some theory to develop

One class of examples that we have in mind

Let Q be a number field (that is, a finite extension of $\mathbb{Q}$), and let R be an integrallly closed subring of Q.
Let G be a finite group, and assume that none of the primes dividing the order of G becomes invertible in R.

AIM: *Describe* countably generated projective modules over $\Lambda = RG$.

STRATEGY: *Glue* the information we can obtain of the family of localizations $\Lambda_{\mathfrak{m}} = R_{\mathfrak{m}}G$ for $\mathfrak{m}$ in the maximal spectrum of R. Each of these localizations, if Q is *nice enough*, is semiperfect.

We need also to keep in mind that $\Lambda_Q = QG$ which is semisimple artinian that is, a finite product of matrices over division rings.

Swan (1960): All finitely generated projective right Λ-modules are locally free.

This implies that they are always isomorphic to a free module, direct sum an ideal of Λ.

Now a non-noetherian motivating situation

Let R be a commutative domain and consider

$$\mathcal{F}(R) = \{\bigoplus_{i\in I} M_i : M_i \text{ f.g. torsion-free } R\text{-module}\}$$

$\mathcal{F}(R)$ is closed under direct sums, but

$$\boxed{\text{When is } \mathcal{F}(R) \text{ closed under direct summands?}}$$

Theorem (R. Álvarez, H., P. Příhoda, 2024)

Let R be a local commutative domain of Krull dimension 1. The following statements are equivalent:

(i) *$\mathcal{F}$ is closed under direct summands. That is, direct summands of direct sums of finitely generated torsion-free modules are direct sum of finitely generated modules;*

(ii) *finitely generated indecomposable torsion-free modules have local endomorphism ring.*

If, in addition, R is Noetherian this is equivalent to R having local integral closure (Příhoda, 2022).

From local to global: Determine projective modules over the endomorphism ring

A commutative domain R has finite character if for any $0 \neq d \in R$, R/dR has only a finite number of maximal ideals. It is h-local if, in addition, any non-zero prime ideal is contained in a unique maximal ideal.

All noetherian domains of Krull dimension 1 have finite character and, hence, are h-local.

Proposition (R. Álvarez, H., P. Příhoda, 2024)

*Let R be a domain of of Krull dimension 1 and of finite character (=h-local). Let X be a finitely generated torsion-free R-module, set $\Lambda = \mathrm{End}_R(X)$ and assume that every direct summand of $R^{(\omega)} \oplus X$ is a direct sum of finitely generated modules. Then, for any maximal ideal $\mathfrak{m}$ of R, $X_\mathfrak{m}$ is a finite sum of modules with local endomorphism ring so $\Lambda_\mathfrak{m}$ is a **semiperfect ring**.*

In this case, Λ is a torsion-free R-algebra and $\Lambda \subseteq \Lambda_Q \cong M_n(Q)$.

(Dress 70's) M any finitely generated right module over a general associative ring S. The category $\mathrm{Add}(M)$, of direct summands of arbitrary number of copies of M, is equivalent to the category of projective right modules over $\Lambda = \mathrm{End}_S(M)$.

Locally semiperfect algebras over domains of finite character

In both situations, we are dealing with the following class of algebras:

Definition

Let R be a commutative domain. Let Λ be an R-algebra, we say that Λ is *locally semiperfect* if $\Lambda_{\mathfrak{m}}$ is semiperfect for any maximal ideal $\mathfrak{m}$ of R.

In additon, in our cases R is h-local, Λ is R-torsion-free and Λ_Q is semisimple artinian.

Monoids of finitely generated projective modules

Λ associative ring with 1,
$V(\Lambda)$ = set of representatives of isomorphism classes of finitely generated projective right Λ modules = set of representatives of isomorphism classes of direct summands of the right module Λ^n for some n
Is a commutative (additive) monoid with the addition induced by the direct sum:

$$[P] + [P'] = [P \oplus P']$$

This monoid satisfies:

(1) Is reduced $[P] + [P'] = 0$ if and only if $P = 0 = P'$.
(2) $[\Lambda]$ is an *order unit*: for any $[P] \in V(\Lambda)$ there exists $n > 0$ and $[P'] \in V(\Lambda)$ such that $[P] + [P'] = n[\Lambda] = [\Lambda^n]$

Theorem

(G. Bergman, 1974) Any commutative monoid satisfying (1) and (2) can be realized as $V(\Lambda)$ for Λ a (semihereditary) ring.

General question: which kind of monoids can be realized as $V(\Lambda)$ for particular classes of rings?

The very nice cases: semisimple artinian rings and semiperfect rings

If Λ is a semisimple artinian ring then $\Lambda \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ and there are $V_1, \dots, V_k$ simple right modules such that any right Λ module is a direct sum of such simple modules in a unique way.

The assignment

$$[P] = r_1[V_1] + \cdots + r_k[V_k] \mapsto (r_1, \dots, r_k)$$

where $P \cong V_1^{r_1} \oplus \cdots \oplus V_k^{r_k}$, induces an isomorphism $\varphi \colon V(\Lambda) \to \mathbb{N}_0^k$ and $\varphi([\Lambda]) = (n_1, \dots, n_k)$

If Λ is a semiperfect ring then $\Lambda/J(\Lambda)$ is semisimple artinian an idempotents lift. So all projective modules are direct sums, in a unique way, of indecomposable cyclic projective module $P_1, \dots, P_k$ and

$$V(\Lambda) \cong V(\Lambda/J(\Lambda)) \cong \mathbb{N}_0^k$$

What about infinitely generated projective modules?

Theorem (I. Kaplansky, 1958)

Any projective module is a direct sum of countably generated projective modules

We can consider $V^*(\Lambda) =$ set of representatives of isomorphism classes of countably generated projective modules

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The assignment

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induces an isomorphism $\varphi\colon V^*(\Lambda) \to (\mathbb{N}_0^*)^k$ where $\mathbb{N}_0^* = \mathbb{N}_0 \cup \{\infty\}$

Similarly, if Λ is semiperfect $V^*(\Lambda) \cong (\mathbb{N}_0^*)^k$

Traces of projective modules

If P is a projective right Λ module then its trace is

$$\mathrm{Tr}(P) = \sum_{f \in \mathrm{Hom}_\Lambda(P,\Lambda)} f(P).$$

Dual basis Lemma implies that traces of projectives are idempotent ideals and $P = P\mathrm{Tr}(P)$. Traces are two-sided ideals

Trace ideals of countably generated projective modules were nicely characterized by Whitehead (1980)

Theorem (Whitehead, 1980)

If I is an idempotent ideal that is finitely generated as a left Λ-module then I is the trace ideal of a countably generated projective right module.
In particular, if Λ is noetherian the idempotent ideals are exactly the ideals that are traces of countably generated projective modules.

G. Puninski found this forgotten result of Whitehead around 2005 and realized how interesting it could be in the study of projective modules.

Easy examples of traces

Basic facts:

- $\{0\}$ and Λ are idempotent ideals and they are the trace of $\{0\}$ and of any non-zero free-module, respectively.
- If $e^2 = e \in \Lambda$ then $\mathrm{Tr}_\Lambda(e\Lambda) = \Lambda e \Lambda = \mathrm{Tr}_\Lambda(\Lambda e)$.
- Trace of a direct sum is the sum of the traces.

If Λ is semisimple artininian, all its ideals are trace ideals

$$\Lambda \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$$

So traces can be parametrized by $\mathcal{P}(\{1, \ldots, k\})$.
There is also such a correspondence if Λ is a semiperfect ring with k indecomposable projective modules.

In the case of $\Lambda = RG$, with G finite group and R suitable Dedekind domain containing $\mathbb{Z}$, and such that no prime divisor of the order of G is invertible in R, then finitely generated projective modules (that are locally free) have trace R.

Roggenkamp (1974): $\{0\}$ and Λ are the only idempotent ideals of Λ if and only if G is solvable.

Lifting projective modules modulo a trace ideal

From now on let $\mathcal{T}(\Lambda)$ denote the set of traces of countably generated projective right Λ-modules.

A consequence of Eilenberg's swindle: P_1, P_2 countably generated projective. Then $\mathrm{Tr}(P_1) = \mathrm{Tr}(P_2)$ if and only if $P_1^{(\omega)} \cong P_2^{(\omega)}$.

Theorem (P. Příhoda (2010), H.- Příhoda (2014))

Let Λ be a ring, and let I be the trace of a projective right Λ-module. Let P' be a projective right Λ/I-module. Then there is a projective right Λ-module X such that $I \subseteq \mathrm{Tr}_\Lambda(X)$, $X/XI \cong P'$ and $\mathrm{Tr}_\Lambda(X)/I = \mathrm{Tr}_{\Lambda/I}(P')$.
If I is an element in $\mathcal{T}(\Lambda)$ and P' is finitely generated, then X can be taken to be countably generated. Such lifting is unique in the sense: if Y is another lifting of P' then

$$X \oplus P^{(\omega)} \cong Y \oplus P^{(\omega)}$$

where P is any projective countably generated projective Λ-module with trace I.

Relatively big projectives

Definition (P. Příhoda, 2010)

Let Λ be a ring, let I be an ideal of Λ which is the trace of a countably generated projective module.
A countably generated projective right Λ-module P is relatively big with respect to I if:

(1) P/PI is finitely generated;

(2) $P \cong P \oplus (P')^{(\omega)}$ where P' is any countably generated projective module with trace ideal I.

Finitely generated projectives are relatively big with respect to 0, and if P is any countably generated projective then $P^{(\omega)}$ is relatively big with respect to its trace.
The lifting of a finitely generated projective module modulo an ideal of $\mathcal{T}(\Lambda)$ is the typical example of relatively big projective module.
All countably generated modules over a semiperfect ring are relatively big

Remark

A relatively big projective module P is determined, up to isomorphism, by I and by P/PI.

The monoid of relatively big projectives

The set of isomorphism classes of these relatively big projectives is a submonoid $B(\Lambda)$ of $V^*(\Lambda)$.

$$B(\Lambda) \cong \bigcup_{I \in \mathcal{T}(\Lambda)} V(\Lambda/I)$$

If $I = \{0\}$ then $V(\Lambda/I) = V(\Lambda)$, for $I = \Lambda$ then $V(\Lambda/I) \cong \{\langle \Lambda^{(\omega)} \rangle\}$.

Theorem (P. Příhoda 2010)

If Λ is semilocal noetherian then $B(\Lambda) = V^*(\Lambda)$.

Theorem (R. Álvarez, H., P. Příhoda, 2025)

If Λ is a right noetherian PI ring (e.g. Λ is a module finite algebra over a commutative noetherian ring) then $B(\Lambda) = V^*(\Lambda)$.

Corollary (Bass, 1963)

If Λ is an indecomposable commutative noetherian ring then all countably generated projective Λ-modules that are not finitely generated are free.

Summarizing: Relatively big projective are not so far of finitely generated ones

In general, for a noetherian ring Λ the monoid of relatively big projectives is determined by finitely generated data:

- The set of idempotent ideals (that coincides with $\mathcal{T}(\Lambda)$).
- $V(\Lambda/I)$ for $I \in \mathcal{T}(\Lambda)$.

If Λ is close enough to a commutative ring (it is PI!) this gives all countably generated projective modules. The advantage is that for PI noetherian rings $\mathcal{T}(\Lambda)$ is finite (Small-Robson, 1976).

From local to global in $\Lambda = RG$

G finite group.

R is a dedekind domain whose field of fractions Q is an algebraic number field and such that no prime dividing $|G|$ is invertible in R.

We assume that Q has enough roots of unity, so that the endomorphism ring of any simple module over QG is Q.

In this situation $\Lambda_\mathfrak{m} = R_\mathfrak{m} G$ is semiperfect for any maximal ideal $\mathfrak{m}$ of R.

Moreover, there is a finite set $\mathcal{S}$ of maximal ideals of R such that for any maximal ideal $\mathfrak{m}$ of R that is not in $\mathcal{S}$, one has that if P is a finitely generated projective $\Lambda_\mathfrak{m}$-module then it is indecomposable if and only if $P \otimes_R Q$ is indecomposable.

From local to global in $\Lambda = RG$: traces

Applying results by Levy and Odenthal (1996), one can "easily" pass from local to global to compute idempotent ideals:

- If I is an idempotent ideal of Λ then $I_{\mathfrak{m}}$ is an idempotent ideal of $\Lambda_{\mathfrak{m}}$ and $I_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} Q = I \otimes_R Q$.
- Conversely, any family $I(\mathfrak{m})$ of idempotent ideal of $\Lambda_{\mathfrak{m}}$ where $\mathfrak{m}$ ranges through all maximal ideals of R, such that $I(\mathfrak{m}) \otimes_R Q = I(\mathfrak{n}) \otimes_R Q$ for any pair of maximal ideals of R, gives a unique idempotent ideal I of Λ such that $I_{\mathfrak{m}} = I(\mathfrak{m})$.

In particular, idempotent ideals are completely determined by their localizations

From local to global in $\Lambda = RG$: $B(\Lambda)$

If I is an idempotent ideal $V(\Lambda/I)$ is determined, up to isomorphism, provided Λ/I is of finite length. But not in general, so by now we are just happy to determine the “genus" of the corresponding relatively big projective.

Let $V_1, \dots, V_k$ be a set of representatives of the simple QG-modules. For each maximal ideal $\mathfrak{m}$ of R fix

$$P_1^{\mathfrak{m}}, \dots, P_{\ell_{\mathfrak{m}}}^{\mathfrak{m}}$$

to be a set of representatives of the indecomposable projective $\Lambda_{\mathfrak{m}}$-modules. Consider a matrix $D^{\mathfrak{m}} = (d_{ij}^{\mathfrak{m}})$ where
$d_{ij}^{\mathfrak{m}}$ = The number of occurrencies of the simple V_i in the decomposition of $P_j^{(m)} \otimes_R Q$

From local to global in $\Lambda = RG$: $B(\Lambda)$

Theorem (R. Álvarez, H., P. Příhoda, 2025)

For each maximal ideal $\mathfrak{m}$ of R pick an element $r_\mathfrak{m} \in (\mathbb{N}_0^*)^{\ell_\mathfrak{m}}$. Then there exists a countably generated projective Λ-module P such that $P_\mathfrak{m}$ corresponds to $r_\mathfrak{m}$ for any $\mathfrak{m}$ if and only if

$$D^\mathfrak{m} r_\mathfrak{m} = D^\mathfrak{n} r_\mathfrak{n}$$

for any pair of maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ of R.

Recall that just a finite number of the matrices $D^\mathfrak{m}$ are different from the identity.
Such matrices are called **decomposition matrices** and can be computed.

Example

In $\Lambda = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]A_5$ there are three non-trivial idempotent ideals. In this case, Λ/I is either a PID or has finite length.
So one determines the countably generated projective modules, that are not finitely generated, up to isomorphism.
This is no longer true for A_6!!

.... but also all projectives over our endomorphism algebras are relatively big!

Keep in mind the example R h-local domain, M finitely generated torsion-free R-module and $\Lambda = \mathrm{End}_R(M)$.

Theorem (R. Álvarez, H., P. Příhoda, 2025)

Let R be an h-local domain with field of fractions Q, and let Λ be a locally semiperfect torsion-free R-algebra such that Λ_Q is a simple artinian ring. Then:

1. *Any trace ideal of Λ is the trace of a finitely generated projective module*
2. *$\mathcal{T}(\Lambda)$ is finite.*
3. *$V^*(\Lambda) = B(\Lambda)$.*
4. *"only a finite number of maximal ideals of R are important."*

This allows to make a description of the genus of countably generated projective modules over such Λ as before.

But now the countably generated projective modules, that are not finitely generated, have the same genus if and only if they are isomorphic.

Some consequences

Theorem (R. Álvarez, H., P. Příhoda, 2025)

Let R be an h-local domain and let M be a non-zero finitely generated torsion-free R-module with endomorphism ring $\Lambda = \mathrm{End}_R(M)$. Consider the following statements,

(i) *Λ is locally semiperfect and if X and Y are indecomposable direct summands of $M_{\mathfrak{m}}$ and $N_{\mathfrak{n}}$ for some $\mathfrak{m} \neq \mathfrak{n}$ maximal ideals of R, then $\mathrm{rank}(X)$ and $\mathrm{rank}(Y)$ are coprime.*

(ii) *Every module in $\mathrm{Add}(M)$ is a direct sum of finitely generated modules.*

Then (i) always implies (ii) and the converse is true provided M is a generator and R has Krull dimension 1.

Corollary (R. Álvarez, H., P. Příhoda, 2024)

Let R be a domain of finite character, and of Krull dimension 1. Assume that $\mathcal{F}$ is closed under direct summands.
Then for all maximal ideals $\mathfrak{m}$ of R, except may be one, any finitely generated ideal of $R_{\mathfrak{m}}$ is two-generated.

Thank you for your kind attention