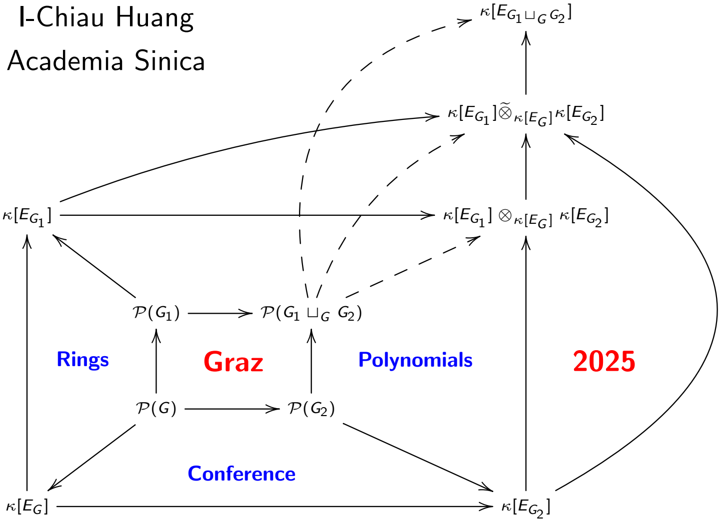
Edge Algebras and their Fibered Sums

I-Chiau Huang

Academia Sinica



This is a joint work with

C.-Y. Jean Chan (Central Michigan University, USA) and
Jung-Chen Liu (National Taiwan Normal University, Taiwan).

Our viewpoint is also developed with

Raheleh Jafari (Kharazmi University, Iran) and
Mee-Kyoung Kim (Sungkyunkwan University, Korea).

absolute viewpoint	relative viewpoint
affine semigroup ring	affine semigroup algebra
edge ring	edge algebra
gluing	fibered sum

Affine Semigroup (intrinsic definition)

An *affine semigroup* is a finitely generated commutative monoid S that satisfies the following intrinsic properties: for any $s, s_1, s_2 \in S$,

- (*cancellative*) if $s + s_1 = s + s_2$, then $s_1 = s_2$;
- (*torsion free*) if $ns_1 = ns_2$ in S for some positive integer n, then $s_1 = s_2$.

Affine Semigroup (characterization)

A finitely generated commutative monoid is an affine semigroup if and only if it can be embedded in $\mathbb{Z}^d$ (equivalently in $\mathbb{Q}^d$) for some positive integer d.

Affine Semigroup Ring

Let κ be a field. With respect to an embedding of S into some $\mathbb{Z}^d$, an element of S is realized as a d-tuple $(i_1, \ldots, i_d)$ of integers. The *affine semigroup ring* $R := \kappa[\mathbf{u}^S]$ is the subring of the Laurent polynomial ring $\kappa[\mathbf{u}, \mathbf{u}^{-1}]$ in the variables $\mathbf{u}$ generated by $u_1^{i_1} \cdots u_d^{i_d}$, where $(i_1, \ldots, i_d) \in S$.

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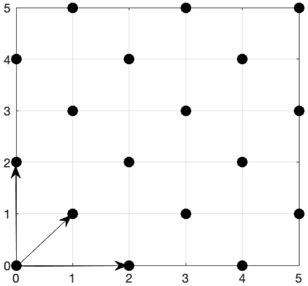
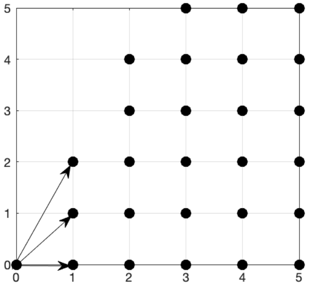
Examples.

$\langle (2,0),(1,1),(0,2)\rangle \subset \mathbb{Z}^2$



$\langle (1,0),(1,1),(1,2)\rangle \subset \mathbb{Z}^2$



The affine semigroup rings $\kappa[u_1^2, u_1u_2, u_2^2]$ and $\kappa[v_1, v_1v_2, v_1v_2^2]$ can be identified via $u_1 = v_1^{1/2}$ and $u_2 = v_1^{1/2}v_2$.

- If S is an affine sub-semigroup of another affine semigroup S', we may consider them as sub-semigroups of some $\mathbb{Z}^d$.

$$S \subset S' \subset \mathbb{Z}^d$$

We say that S shares the embedding of S'.

- The semigroup ring $R' := \kappa[\mathbf{u}^{S'}]$ is an algebra over the coefficient ring $R := \kappa[\mathbf{u}^S]$. We use R'/R to denote the *affine semigroup algebra* R' over R.
- Given a numerical semigroup ring $\kappa[\![\mathbf{u}^S]\!]$, there arise two numerical semigroup algebras $\kappa[\![\mathbf{u}]\!]/\kappa[\![\mathbf{u}^S]\!]$ and $\kappa[\![\mathbf{u}^S]\!]/\kappa[\![\mathbf{u}^s]\!]$, where $0 \neq s \in S$.
  - Algebraic properties of $\kappa[\![\mathbf{u}^S]\!]$ are properties of the flat numerical semigroup algebra $\kappa[\![\mathbf{u}^S]\!]/\kappa[\![\mathbf{u}^s]\!]$.
  - Arithemetic properties of $\kappa[\![\mathbf{u}^S]\!]$ are properties of the coefficient ring of the equi-gcd numerical semigroup algebra $\kappa[\![\mathbf{u}]\!]/\kappa[\![\mathbf{u}^S]\!]$.

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- Let G be a finite simple graph with vertices $t_1, \ldots, t_n$.
- An edge of G connecting t_i and t_j is denoted by $t_i t_j \in \kappa[t_1, \ldots, t_n]$.
- The edge ring $\kappa[E_G]$ of G is the subring of $\kappa[t_1, \ldots, t_n]$ generated as a κ-algebra by the edges of G.
- The edge rings of the graphs

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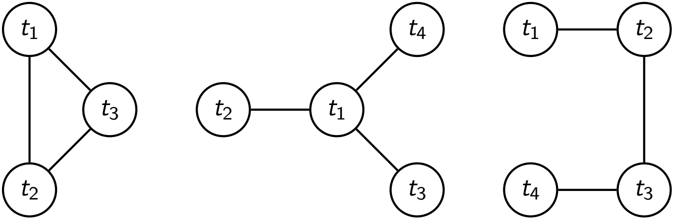

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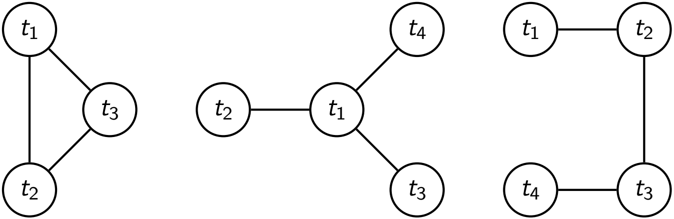

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Toric ideal

Let G be a finite simple graph. We consider the polynomial ring $\mathcal{P}(G)$ over a field κ, whose variables correspond to edges of G. Let

$$\mathbf{e}\colon \mathcal{P}(G) \to \kappa[E_G]$$

be the homomorphism of κ-algebras sending each variable to its corresponding edge. The kernel of $\mathbf{e}$ is called the *toric ideal* of G and is denote by I_G.

Splitting

A toric ideal I_G is *splittable* if there exist subgraphs G_1 and G_2 of G such that $I_G = I_{G_1} + I_{G_2}$ and $I_G \neq I_{G_i}$ for $i = 1, 2$.

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Gluing

Let T_1 and T_2 be numerical semigroups.

Let $a \in T_1$ and $b \in T_2$ be relatively prime numbers such that a (resp. b) is not a minimal generator of T_1 (resp. T_2).

The numerical semigroup $bT_1 + aT_2$ is called a *gluing* of T_1 and T_2.

- Gluing is a fibered sum in the category of numerical semigroups.

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In the category of finite simple graphs, fibered sum exists.

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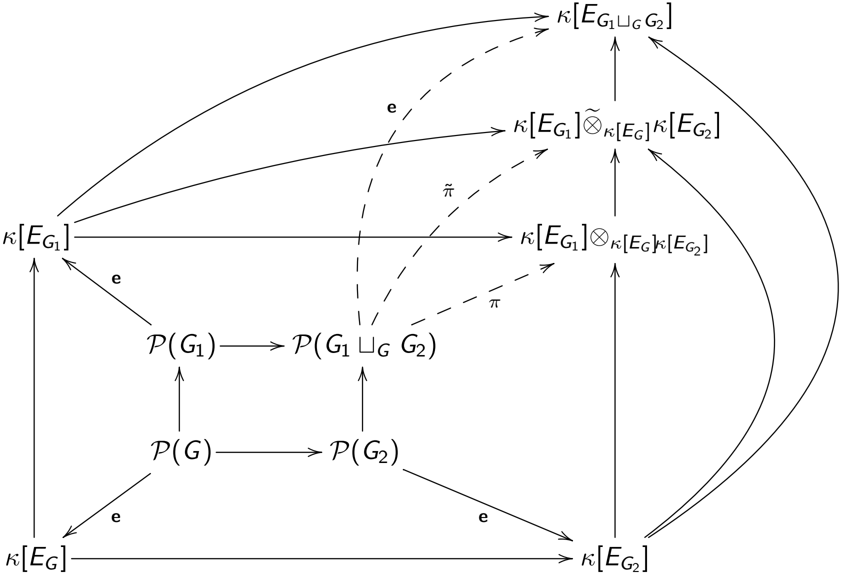
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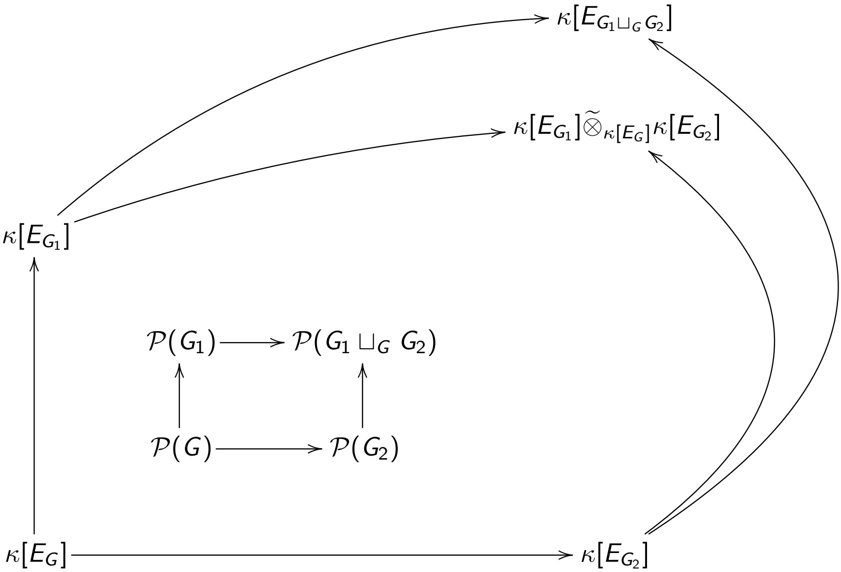
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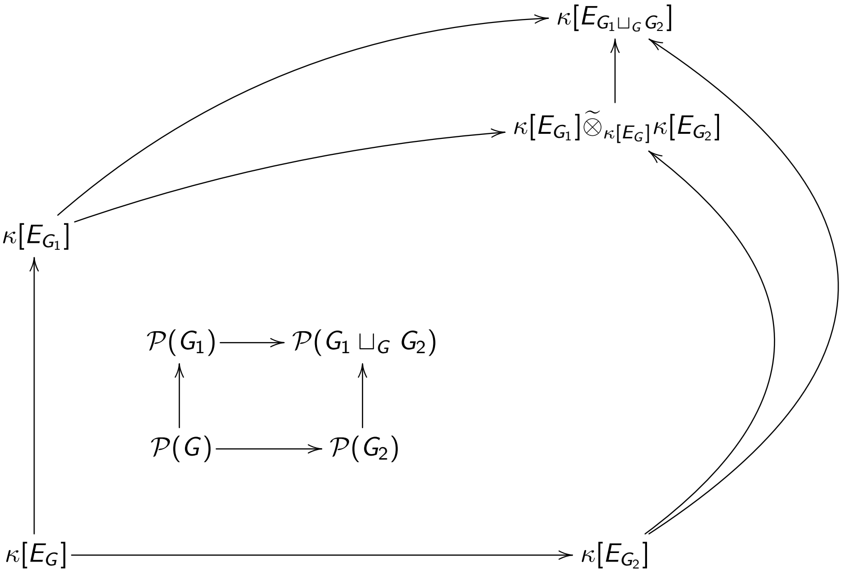
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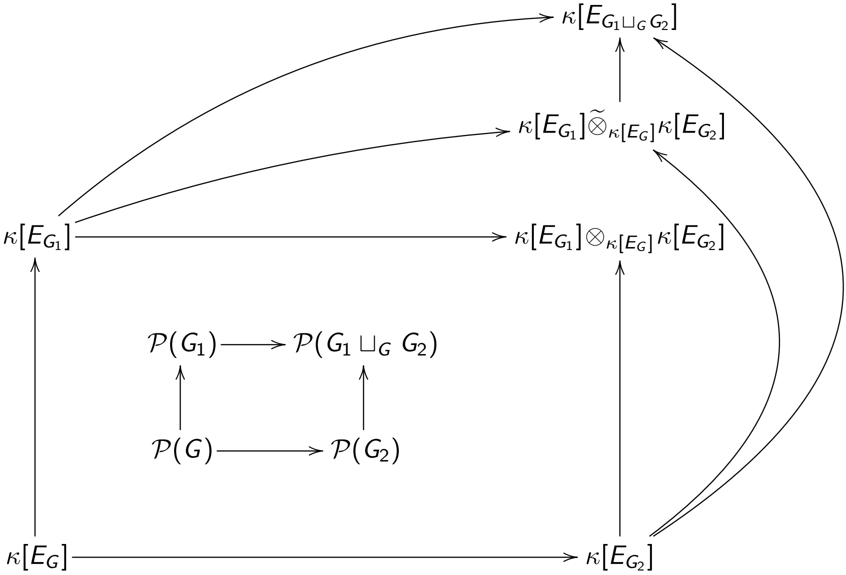
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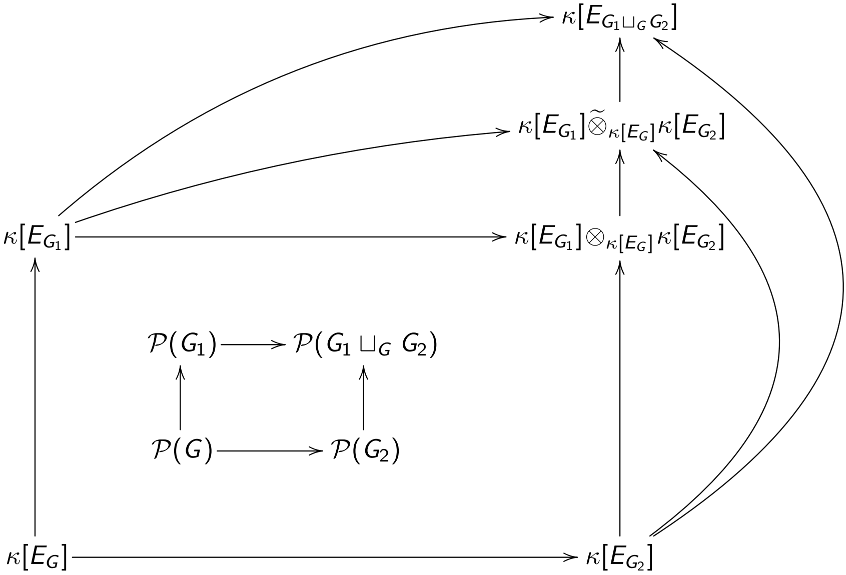

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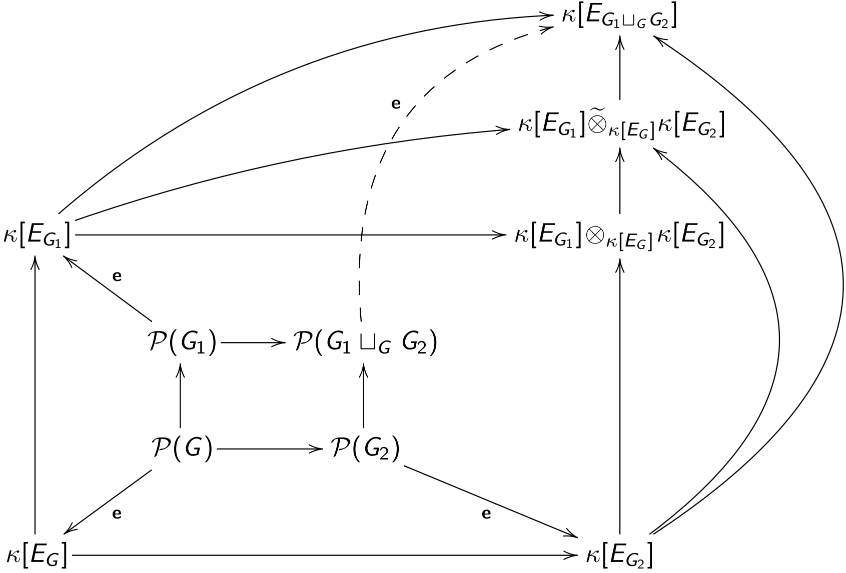

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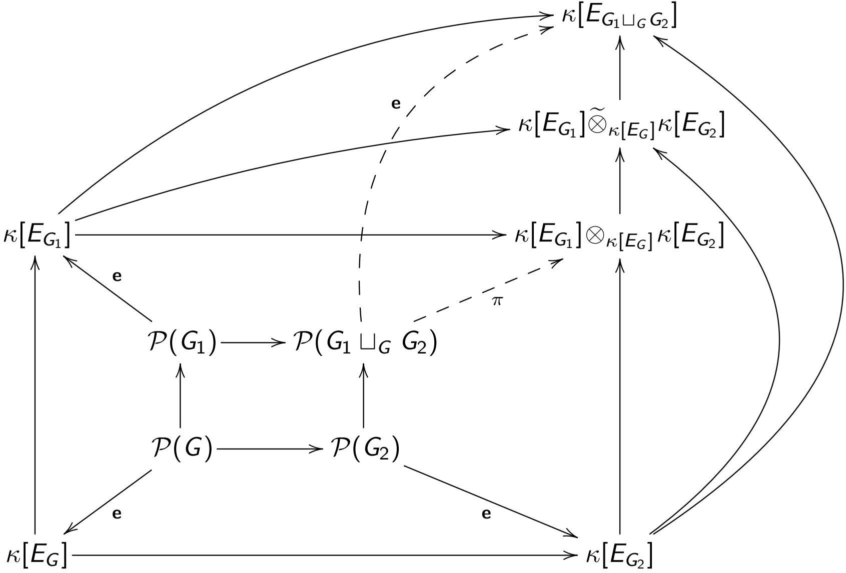
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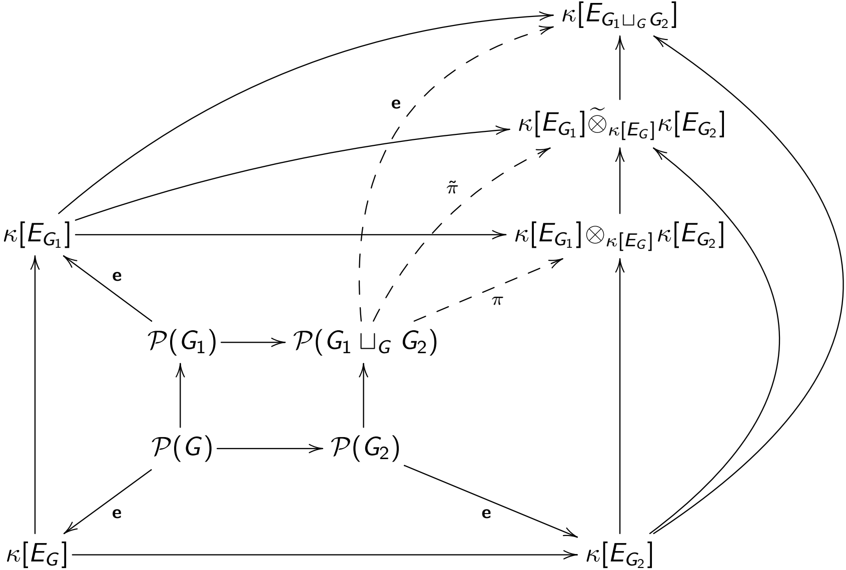

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$\mathcal{P}(G_1)$
$\mathcal{P}(G_1 \sqcup_G G_2)$
$\mathcal{P}(G)$
$\mathcal{P}(G_2)$
$\kappa[E_G]$
$\kappa[E_{G_2}]$
e
$\tilde{\pi}$
π

$$
\begin{array}{ccccc}
 & & \mathcal{P}(G_1 \sqcup_G G_2) & & \\
 & \swarrow^{\pi} & \downarrow{\scriptstyle \tilde{\pi}} & \searrow^{\mathbf{e}} & \\
\kappa[E_{G_1}] \otimes_{\kappa[E_G]} \kappa[E_{G_2}] & \longrightarrow & \kappa[E_{G_1}] \widetilde{\otimes}_{\kappa[E_G]} \kappa[E_{G_2}] & \longrightarrow & \kappa[E_{G_1 \sqcup_G G_2}]
\end{array}
$$

Proposition

Let $G \to G_1$ and $G \to G_2$ be inclusions of finite simple graphs. Then $I_{G_1 \sqcup_G G_2} = I_{G_1} + I_{G_2}$ if and only if the canonical surjection

$$\kappa[E_{G_1}] \otimes_{\kappa[E_G]} \kappa[E_{G_2}] \to \kappa[E_{G_1 \sqcup_G G_2}]$$

obtained by the universal property of the tensor product is an isomorphism.

$$
\begin{array}{ccccc}
 & & \mathcal{P}(G_1 \sqcup_G G_2) & & \\
 & \swarrow^{\pi} & \downarrow^{\tilde{\pi}} & \searrow^{\mathbf{e}} & \\
\kappa[E_{G_1}] \otimes_{\kappa[E_G]} \kappa[E_{G_2}] & \longrightarrow & \kappa[E_{G_1}] \widetilde{\otimes}_{\kappa[E_G]} \kappa[E_{G_2}] & \longrightarrow & \kappa[E_{G_1 \sqcup_G G_2}]
\end{array}
$$

Theorem

Let G_1 and G_2 be subgraphs of a finite simple graph such that $G := G_1 \cap G_2$ is a K_2. Then $\kappa[E_{G_1}] \otimes_{\kappa[E_G]} \kappa[E_{G_2}]$ is torsion free.

$$\begin{array}{ccccc} & & \mathcal{P}(G_1 \sqcup_G G_2) & & \\ & \swarrow^{\pi} & \downarrow{\scriptstyle\tilde{\pi}} & \searrow^{\mathbf{e}} & \\ \kappa[E_{G_1}] \otimes_{\kappa[E_G]} \kappa[E_{G_2}] & \longrightarrow & \kappa[E_{G_1}] \widetilde{\otimes}_{\kappa[E_G]} \kappa[E_{G_2}] & \longrightarrow & \kappa[E_{G_1 \sqcup_G G_2}] \end{array}$$

Proposition

Let G_1 and G_2 be subgraphs of a finite simple graph such that $G := G_1 \cap G_2 \simeq K_2$. The canonical map

$$\kappa[E_{G_1}] \widetilde{\otimes}_{\kappa[E_G]} \kappa[E_{G_2}] \to \kappa[E_{G_1 \sqcup_G G_2}]$$

is an isomorphism if and only if G_1 or G_2 is bipartite.

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THANK YOU!