Power graphs of semigroups of homogeneous elements of graded rings

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Let S be a groupoid (magma) with a multiplicative operation. S is said to be *power-associative* if for every $x \in S$ a subgroupoid $\langle x \rangle$ of S, generated by x, is a semigroup.

[1] J. Abawajy, A. Kelarev, M. Chowdhury, Power Graphs: A Survey, *Electron. J. Graph Theory Appl.* **1** (2) (2013) 125–147.

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The *directed power graph*[1] $\overrightarrow{\mathcal{G}}(S)$ of S is a directed graph with the set of vertices S, and with edges (x, y) such that $x \neq y$ and y is a power of x.

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The *undirected power graph* $\mathcal{G}(S)$ of S is the underlying undirected graph of $\overrightarrow{\mathcal{G}}(S)$, that is, S is its set of vertices and two distinct vertices are adjacent if and only if one is a power of the other.

[1] J. Abawajy, A. Kelarev, M. Chowdhury, Power Graphs: A Survey, *Electron. J. Graph Theory Appl.* **1** (2) (2013) 125–147.

The directed power graphs of groups are defined in [2], and the directed power graphs of semigroups are first introduced and studied in [3], [4], [5]. In these papers, the term 'power graph' is used for the directed power graph, which covers the notion of the undirected power graph as the underlying undirected graph of the directed power graph. The undirected power graphs are the main object of study in [6], where they are also briefly referred to as the power graphs. In this talk, we use the brief term 'power graph' for an 'undirected power graph'.

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[2] A. V. Kelarev, S. J. Quinn, A combinatorial property and power graphs of groups, In: Contributions to General Algebra, vol. 12, Springer-Verlag, 2000, pp. 229–235.

[3] A. V. Kelarev, S. J. Quinn, Directed graphs and combinatorial properties of semigroups, *J. Algebra* **251** (2002) 16–26.

[4] A. V. Kelarev, S. J. Quinn, A combinatorial property and power graphs of semigroups, *Comment. Math. Univ. Carolinae* **45** (2004) 1–7.

[5] A. V. Kelarev, S. J. Quinn, R. Smolikova, Power graphs and semigroups of matrices, *Bull. Austral. Math. Soc.* **63** (2001) 341–344.

[6] I. Chakrabarty, S. Ghosh, M. K. Sen, Undirected power graphs of semigroups, *Semigroup Forum* **78** (2009) 410–426.

Definition[7], [8], [9]:

Let R be a ring, and Δ a partial groupoid (magma), that is, a set with a partial binary operation. Also, let $\{R_\delta\}_{\delta\in\Delta}$ be a family of additive subgroups of R. We say that $R = \bigoplus_{\delta\in\Delta} R_\delta$ is Δ-*graded* and R *induces* Δ (or R is an Δ-*graded ring inducing* Δ) if the following two conditions hold:

i) $R_\xi R_\eta \subseteq R_{\xi\eta}$ whenever $\xi\eta$ is defined;

ii) $R_\xi R_\eta \neq 0$ implies that the product $\xi\eta$ is defined.

[7] A. V. Kelarev, On groupoid graded rings, *J. Algebra* **178** (1995) 391–399.

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i) $R_\xi R_\eta \subseteq R_{\xi\eta}$ whenever $\xi\eta$ is defined;

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It is said to be *strongly graded* if the equality holds in *i*).

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We say that Δ is *cancellative* if each of the equalities $\delta\xi = \delta\eta$ or $\xi\delta = \eta\delta$, provided $\delta\xi$, $\delta\eta$, $\xi\delta$, $\eta\delta$ exist, implies $\xi = \eta$, for δ, ξ, $\eta \in \Delta$.

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If A is a subset of R, then by A^* we denote the set $A \setminus \{0\}$.

If $R = \bigoplus_{\delta \in \Delta} R_\delta$ is a Δ-graded ring inducing Δ, then we say that R satisfies condition $(\diamond)$ if

$$(\forall \delta, \gamma \in \Delta)\ (\delta \notin E(\Delta) \wedge \gamma \notin E(\Delta) \wedge \delta\gamma \in E(\Delta)^*) \Rightarrow R_\delta R_\gamma = \{0\}.$$

Problem[10]:

Let G be a group with identity e, and let $R = \bigoplus_{g \in G} R_g$ be a crossed product or a group-graded ring. Reduce the various parameters of the graphs $\overrightarrow{\mathcal{G}}(H_R)$ and $\mathcal{G}(H_R)$ to the corresponding properties of the subring R_e and the group G.

[10] J. Abawajy, A. Kelarev, M. Chowdhury, Power Graphs: A Survey, *Electron. J. Graph Theory Appl.* **1**(2) (2013) 125–147.

Proposition[11]:

Let S be a semigroup. Then, if the power graph $\mathcal{G}(S)$ is connected, S contains at most one idempotent element.

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Lemma[12]:
Let S be a semigroup with a nonempty set of idempotent elements $E(S)$. If $e \in E(S)$, and if $\mathcal{G}(S)$ is connected, then s and e are adjacent in $\mathcal{G}(S)$ for every $e \neq s \in S$.

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Proposition[12]:
Let $R = \bigoplus_{s \in S} R_s$ be an S-graded ring inducing S. Then the power graph $\mathcal{G}(H_R)$ is connected if and only if R is a graded-nil ring.

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Let S be a semigroup with zero. Then, $\mathcal{G}(S)$ is connected if and only if S is nil.

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Theorem[13]:

Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S, and let S be cancellative. Moreover, let us assume that R has a unique maximal right ideal, and that it satisfies condition $(\diamond)$[14]. Then, the power graph $\mathcal{G}(H_R)$ is connected if and only if the power graph $\mathcal{G}(R_e)$ of the multiplicative semigroup R_e is connected for every idempotent element $e \in S$.

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Corollary:

Let G be a group with identity e, and let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring. Moreover, let us assume that R has a unique maximal right ideal, and that it satisfies condition $(\diamond)$. Then, the power graph $\mathcal{G}(H_R)$ is connected if and only if the power graph $\mathcal{G}(R_e)$ of the multiplicative semigroup R_e is connected.

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Theorem[15]:

Let S be an epigroup with a finite number of idempotent elements, and let $R = \bigoplus_{s \in S} R_s$ be an S-graded ring. Moreover, let us assume that for every nontrivial subgroup G of S, a nonzero ring $R_G = \bigoplus_{s \in G} R_s$ has a unique maximal right ideal, and that R_G satisfies condition $(\diamond)$. Then the power graph $\mathcal{G}(H_R)$ is connected if and only if the power graph $\mathcal{G}(R_e)$ of the multiplicative semigroup R_e is connected for every idempotent element $e \in S$.

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A homogeneous right ideal I of R, that is, an ideal of R such that $I = \bigoplus_{s \in S} I \cap R_s$, is said to be a *graded modular right ideal* if there exists a homogeneous element $u \in R$, called a *left unity modulo* I, such that $ux - x \in I$ for every homogeneous element $x \in R$.

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If S is cancellative, and I a proper graded modular right ideal of R, then all of the left unities modulo I are of the same degree, which is an idempotent element of S, called the *degree of* I.

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If S is cancellative, the *graded Jacobson radical* [16] *of* R, denoted by $J^g(R)$, is equal to the intersection of all maximal graded modular right ideals of R.

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Theorem[17], [18]:
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Theorem[17], [18]:
Let $R = \bigoplus_{s \in S} R_s$ be an S-graded ring inducing S. If S is cancellative, then:

(i) The mapping $I \mapsto I \cap R_e$ defines a one-to-one correspondence between the set of all maximal graded modular right ideals of R of degree e and the set of all maximal modular right ideals of the ring R_e. In particular, $J^g(R) \cap R_e = J(R_e)$ for every idempotent element $e \in S$;

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(ii) The largest homogeneous ideal $J_l^g(R)$ of R, contained in $J(R)$, is contained in $J^g(R)$. If $x \in J_l^g(R)$ is a homogeneous element, then there exists an idempotent element $e \in S$ and an integer n such that $x^n \in R_e$.

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Let $R = \bigoplus_{\delta \in \Delta} R_\delta$ be a Δ-graded ring inducing Δ. If Δ is cancellative, then R is with unity 1 if and only if:

a) R_ε is a ring with unity 1_ε for all $\varepsilon \in E(\Delta)$;

b) for every $x \in H_R$ there exist $\xi, \eta \in E(\Delta)$ such that $1_\xi x = x = x 1_\eta$;

c) $E(\Delta)^*$ is finite,

in which case $1 = \sum_{\varepsilon \in E(\Delta)} 1_\varepsilon$.
If Δ is cancellative, and if R satisfies a) and b), then R is said to be *pseudo-unitary*[19].

[19] E. Halberstadt, Théorie artinienne homogène des anneaux gradués à grades non commutatifs réguliers, PhD Thesis, University Piere and Marie Curie, Paris, France (1971)

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Problem

Investigate connectedness of $\mathcal{G}^\circ(H_R)$ in terms of the properties of R_ε and Δ, where $\varepsilon \in E(\Delta)^*$!

If S is a semigroup, an ideal P of S is said to be *strongly prime*[20] if there exists a subsemigroup M of S such that P is the largest ideal of S not meeting M, that is, $P = \{a \in S \mid S^1 a S^1 \cap M = \emptyset\}$. M is said to be a subsemigroup associated with P.

By $\mathcal{P}(S)$ we denote the set of all strongly prime ideals of S, and by $\mathcal{A}(P)$ the set of all subsemigroups of S associated with $P \in \mathcal{P}(S)$.

[20] R. Slover, The radicals of a semigroup, *Trans. Amer. Math. Soc.* **204** (1975) 179–195.

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Theorem[20]:

A semigroup S with zero is nil if and only if $\mathcal{P}(S) = \emptyset$.

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Theorem[20]:

If S is a semigroup with zero, and $N(S)$ is the nil radical of S, then $N(S) = \bigcap_{P \in \mathcal{P}(S)} P$.

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Theorem[21]:
Let $R = \bigoplus_{\delta \in \Delta} R_\delta$ be a pseudo-unitary Δ-graded ring inducing Δ. Moreover, let us assume that R satisfies condition $(\diamond)$[22]. Then the following statements are equivalent:

i) $\mathcal{G}^\circ(H_R)$ is connected;

ii) The following conditions are satisfied:

   a) $(\forall P \in \mathcal{P}(H_R))(\forall M \in \mathcal{A}(P))\, M \subseteq \bigcup_{\varepsilon \in E(\Delta)^*} R_\varepsilon^*$;

   b) R_ε is ring isomorphic to the two-element field $\mathbb{F}_2$ for every $\varepsilon \in E(\Delta)^*$;

iii) $H_R \setminus I(H_R)$ is a nil semigroup.

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[21] E. Ilić-Georgijević, On the power graphs of semigroups of homogeneous elements of graded rings, *Ricerche Mat.* **74** (3) (2024) 1401–1412.

[22] essential

Definition[23]:

Let S be a semigroup with zero 0, and let $\{S_\delta\}_{\delta\in\Delta}$ be a family of subsets of S, called the *components* of S, indexed by a partial groupoid (magma) Δ, such that $S = \bigcup_{\delta\in\Delta} S_\delta$. The semigroup S is said to be a *homogeneous semigroup* or a *graded semigroup*, and *graded by* Δ, if:

i) $S_\xi \cap S_\eta = \{0\}$ for all distinct $\xi, \eta \in \Delta$;

ii) $S_\xi S_\eta \subseteq S_{\xi\eta}$ whenever $\xi\eta$ is defined;

iii) $S_\xi S_\eta \neq \{0\}$ implies that the product $\xi\eta$ is defined.

Graded semigroups are also studied in [24], where Δ is a group.

[23] E. Ilić-Georgijević, On the Jacobson and simple radicals of semigroups, *Filomat* **32** (7) (2018) 2577–2582.

[24] R. Hazrat, Z. Mesyan, Graded semigroups, *Israel J. Math.* **253** (2023) 249–319.

Definition[25]:

A homogeneous semigroup $S = \bigcup_{\delta \in \Delta} S_\delta$ is said to be *pseudo-unitary* if it moreover satisfies the following axioms:

iv) for every idempotent element $\varepsilon \in \Delta$, the subsemigroup S_ε is with identity 1_ε;

v) for every $x \in S$ there exist idempotent elements $\xi, \eta \in \Delta$ such that $1_\xi x = x = x 1_\eta$;

vi) $1_\xi 1_\eta = 1_{\xi\eta}$ whenever $\xi\eta \in \Delta$ is an idempotent element, where ξ, η are idempotent elements of Δ.

[25] E. Ilić-Georgijević, A description of the Cayley graphs of homogeneous semigroups, *Commun. Algebra* **48** (12) (2020) 5203–5214.

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We say that a homogeneous semigroup $S = \bigcup_{\delta \in \Delta} S_\delta$ satisfies condition $(*)$ if

$$(\forall \delta, \gamma \in \Delta)\ (\delta \notin E(\Delta) \wedge \gamma \notin E(\Delta) \wedge \delta\gamma \in E(\Delta)^*) \Rightarrow S_\delta S_\gamma = \{0\}.$$

Theorem[26]:

Let $S = \bigcup_{\delta \in \Delta} S_\delta$ be a pseudo-unitary homogeneous semigroup with zero 0, where Δ is cancellative, and let $\varepsilon \in E(\Delta)^*$. Moreover, let us assume that S satisfies $(*)$, and that $(\forall P \in \mathcal{P}(S))(\forall M \in \mathcal{A}(P))\, M \subseteq \bigcup_{\varepsilon \in E(\Delta)^*} S_\varepsilon^*$. Then $N(S) \cap S_\varepsilon = N(S_\varepsilon)$.

[26] E. Ilić-Georgijević, On the power graphs of semigroups of homogeneous elements of graded rings, *Ricerche Mat.* **74** (3) (2024) 1401–1412.

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Lemma[26]:

Let $S = \bigcup_{\delta\in\Delta} S_\delta$ be a pseudo-unitary homogeneous semigroup with zero 0, where Δ is cancellative, and let $P \in \mathcal{P}(S)$ with an associated subsemigroup M. Then:

i) $0 \notin M$;

ii) there exists a unique $\varepsilon \in E(\Delta)^*$ such that $1_\varepsilon x = x = x 1_\varepsilon$ for all $x \in M$. Moreover, $1_\varepsilon \notin P$, and so, $P = \{a \in S \mid SaS \cap M^1 = \emptyset\}$, where $M^1 = M \cup \{1_\varepsilon\}$. This idempotent element ε is called the *degree of M*.

[26] E. Ilić-Georgijević, On the power graphs of semigroups of homogeneous elements of graded rings, *Ricerche Mat.* **74** (3) (2024) 1401–1412.

Theorem[27]:

Let $S = \bigcup_{\delta \in \Delta} S_\delta$ be a pseudo-unitary homogeneous semigroup with zero 0, where Δ is cancellative, and let $\varepsilon \in E(\Delta)^*$. Moreover, let us assume that S satisfies $(*)$. If $P \in \mathcal{P}(S)$ with an associated subsemigroup of degree ε, then $P_\varepsilon := P \cap S_\varepsilon \in \mathcal{P}(S_\varepsilon)$. If $Q \in \mathcal{P}(S_\varepsilon)$, then

$$\hat{Q} := \{a \in S \mid SaS \cap S_\varepsilon \subseteq Q\} \in \mathcal{P}(S),$$

with an associated subsemigroup of degree ε. If, moreover,

$$(\forall P \in \mathcal{P}(S))(\forall M \in \mathcal{A}(P))\, M \subseteq \bigcup_{\varepsilon \in E(\Delta)^*} S_\varepsilon^*,$$

this gives a one-to-one correspondence between the set of all strongly prime ideals of S, with associated subsemigroups of degree ε, and the set of all strongly prime ideals of S_ε.

[27] E. Ilić-Georgijević, On the power graphs of semigroups of homogeneous elements of graded rings, *Ricerche Mat.* **74** (3) (2024) 1401–1412.

Theorem[28]:

Let $S = \bigcup_{\delta \in \Delta} S_\delta$ be a pseudo-unitary homogeneous semigroup with zero 0, and let Δ be cancellative. Moreover, let us assume that S satisfies $(*)$. Then the following statements are equivalent:

i) $\mathcal{G}^\circ(S)$ is connected;

ii) The following conditions are satisfied:

  a) $(\forall P \in \mathcal{P}(S))(\forall M \in \mathcal{A}(P))\, M \subseteq \bigcup_{\varepsilon \in E(\Delta)^*} S_\varepsilon^*$;

  b) $\mathcal{G}^\circ(S_\varepsilon)$ is connected for every $\varepsilon \in E(\Delta)^*$;

iii) $S \setminus I(S)$ is a nil homogeneous semigroup.

[28] E. Ilić-Georgijević, On the power graphs of semigroups of homogeneous elements of graded rings, *Ricerche Mat.* **74** (3) (2024) 1401–1412.

Let $R = \bigoplus_{s \in S} R_s$ be an S-graded ring inducing S.

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We define $\mathcal{G}^*(H_R)$ as the graph obtained from $\mathcal{G}(H_R)$ by removing 0 and its incident edges.
Let us call $\mathcal{G}^*(H_R)$ the "star power graph" of H_R.

Problem

Investigate connectedness of $\mathcal{G}^*(H_R)$ in terms of the properties of R_e and S, where $e \in E(S)^*$!

Theorem[29]:

Let $R = \bigoplus_{s \in S} R_s$ be a pseudo-unitary S-graded ring inducing S. Then $\mathcal{G}^*(H_R)$ is connected if and only if the following conditions are satisfied:

i) $\mathrm{supp}(R)$ is a periodic group with identity e;

ii) The graph $\mathcal{G}^*(R_e)$ of a multiplicative semigroup R_e is connected;

iii) R is a graded division ring, and so, R_e is a division ring.

[29] E. Ilić-Georgijević, On the power graphs of semigroups of homogeneous elements of graded semisimple Artinian rings, *Commun. Algebra* **52** (11) (2024) 4961–4972.

Theorem[30]:

Let S be a groupoid with zero 0, which is 0-cancellative and which is torsion-free, that is, $s^n = t^n$ implies $s = t$ for all $s, t \in S$, and all positive integers n. Also, let $R = \bigoplus_{s \in S} R_s$ be a contracted S-graded ring which is semisimple Artinian. Then, if $\mathcal{G}^*(R_e)$ is connected for every $e \in E(S)^*$, then the connected components of $\mathcal{G}^*(H_R)$ are precisely the graphs $\mathcal{G}^*(R_e)$, where e runs through $E(S)^*$.

[30] E. Ilić-Georgijević, On the power graphs of semigroups of homogeneous elements of graded semisimple Artinian rings, *Commun. Algebra* **52** (11) (2024) 4961–4972.

Generalizations of Theorem 9.6.1 and Corollary 9.6.3 from [31].

[31] C. Năstăsescu, F. Van Oystaeyen, Methods of Graded Rings, Lecture Notes in Mathematics, Vol. 1836, Springer, Berlin, Heidelberg (2004)

[32] E. Ilić-Georgijević, On the power graphs of semigroups of homogeneous elements of graded semisimple Artinian rings, *Commun. Algebra* **52** (11) (2024) 4961–4972.

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Theorem[32]:

Let S be a groupoid with zero 0, which is torsion-free and 0-cancellative, and let $R = \bigoplus_{s \in S} R_s$ be a contracted S-graded ring with unity. If R is a right Artinian ring, then the support of R is finite.

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Theorem[32]:

Let S be a groupoid with zero 0, which is torsion-free and 0-cancellative, and let $R = \bigoplus_{s \in S} R_s$ be a contracted S-graded ring. If R is semisimple Artinian, then $\mathrm{supp}(R) = E(S)^*$ if R_e is a simple Artinian ring for every $e \in E(S)^*$.

[31] C. Năstăsescu, F. Van Oystaeyen, Methods of Graded Rings, Lecture Notes in Mathematics, Vol. 1836, Springer, Berlin, Heidelberg (2004)

[32] E. Ilić-Georgijević, On the power graphs of semigroups of homogeneous elements of graded semisimple Artinian rings, *Commun. Algebra* **52** (11) (2024) 4961–4972.

Theorem[33]:

Let S be a groupoid with zero 0, which is cancellative and torsion-free. Also, let $E(S)$ be the set of all idempotent elements of S, and let $R = \bigoplus_{s \in S} R_s$ be a pseudo-unitary S-graded ring with finite support, $R_0 = 0$, and which is semiprimary, that is, $R/J(R)$ is semisimple Artinian and $J(R)$ is nilpotent. Then, $J(R) = J^g(R)$. Moreover, if $\mathcal{G}^*(R_e/J(R_e))$ is connected for every $e \in E(S)^* = E(S) \cap \mathrm{supp}(R)$, then the connected components of $\mathcal{G}^*(H_{R/J(R)})$ are precisely the graphs $\mathcal{G}^*(R_e/J(R_e))$, where e runs through $E(S)^*$.

[33] E. Ilić-Georgijević, On the Jacobson radical of graded rings and related power graphs, preprint.

Problem[34]:
Let S be a cancellative periodic partial groupoid such that all subgroups of S are finite and, moreover, their orders have a finite least common multiple. Suppose that R is an S-graded ring inducing S. Is it true that, for each $r \in J(R)$, there exists $n > 0$ such that all homogeneous components of nr belong to $J(R)$?

[34] A. V. Kelarev, *Ring constructions and applications*, Series in Algebra, vol. 9, (World Scientific, New Jersey, London, Singapore, Hong Kong, 2002).

Theorem[35]:

Let S be a cancellative commutative periodic partial groupoid such that all subgroups of S are finite and, moreover, their orders have a finite least common multiple n. Suppose that R is a pseudo-unitary strongly S-graded ring inducing S. Then, for each $r \in J(R)$, all homogeneous components of nr belong to $J(R)$. If, in particular, R is a pseudo-unitary strongly S-graded algebra inducing S over a field of characteristic zero, then $J(R)$ is homogeneous and, moreover, $J(R) = J^g(R)$.

[35] E. Ilić-Georgijević, On the Jacobson radical of graded rings and related power graphs, preprint.

Theorem[36]:

Let S be a cancellative commutative periodic partial groupoid such that all subgroups of S are finite and, moreover, their orders have a finite least common multiple. Suppose that R is a pseudo-unitary strongly S-graded algebra inducing S, over a field of characteristic zero. Moreover, for every $e \in E(S)$, let $K_e = \{s \in S \mid (\exists n \in \mathbb{N})\ s^n = e\}$. Then:

1. $J(R) = J^g(R)$;
2. $T = \mathrm{supp}(R/J(R)) \cup \{0\}$ is a semigroup, and the connected components of the power graph $\mathcal{G}(T)$ are precisely the subgraphs of $\mathcal{G}(T)$, induced by $K_e \cap T$ $(e \in E(S))$;
3. $R_T^e = \bigoplus_{s \in K_e \cap T} R_s$ is a homogeneous subring of R for every $e \in E(S)$;
4. If the graphs $\mathcal{G}^*(R_e/J(R_e))$ are connected for every $e \in E(S)^*$, then the connected components of $\mathcal{G}^*(H_{R/J(R)})$ are precisely the graphs $\mathcal{G}^*(H_{R_T^e/J(R_T^e)})$ $(e \in E(S)^*)$.

[36] E. Ilić-Georgijević, On the Jacobson radical of graded rings and related power graphs, preprint.