On local log-regular rings

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Log-regularity was introduced by Kazuya Kato to establish the theory of toric varieties without bases ([Kat94]).

Definition

$(R, \mathcal{Q}, \alpha)$ is a **log ring**.
$\overset{\text{def}}{\Leftrightarrow}$ It consists of

- R a commutative ring.
- $\mathcal{Q}$ a commutative monoid.
- $\alpha : \mathcal{Q} \to R \ (0 \mapsto 1)$ a monoid homomorphism.

Definition

A log ring $(R, \mathcal{Q}, \alpha)$ is **local**
$\overset{\text{def}}{\Leftrightarrow}$
(1) R is a local ring
(2) $\alpha^{-1}(R^\times) = \mathcal{Q}^*$ where
$\mathcal{Q}^* := \{p \in \mathcal{Q} \mid \text{there exists } q \in \mathcal{Q} \text{ s.t. } p + q = 0\}$.

Definition (K. Kato [Kato94])

$(R, \mathcal{Q}, \alpha)$: a local log ring where

$$(*) \begin{cases} \bullet\ R \text{ is Noetherian, and} \\ \bullet\ \mathcal{Q}_{\text{red}} \text{ is finitely generated, cancellative, root closed.} \end{cases}$$

Set $I_\alpha := \langle \alpha(q) \in R \mid q \in \mathcal{Q} \backslash \mathcal{Q}^* \rangle$.

Then $(R, \mathcal{Q}, \alpha)$ is a **local log-regular ring** if it satisfies

1. R/I_α is a regular local ring.
2. $\dim R = \dim R/I_\alpha + \dim \mathcal{Q}$.

Example

A : a regular domain, $\mathcal{Q}$: a f.g., cancellative, root closed monoid.
$R := A[\mathcal{Q}]$, $\mathfrak{p} \in \operatorname{Spec}(R)$, $P := \mathfrak{p} \cap \mathcal{Q}$, $\iota : \mathcal{Q} \hookrightarrow R$

Then $(R_\mathfrak{p}, \mathcal{Q}_P, \iota_\mathfrak{p})$ is a local log-regular ring where

- $\mathcal{Q}_P$ is a localization of $\mathcal{Q}$ at P,
- $\iota_\mathfrak{p} : \mathcal{Q}_P \to R_\mathfrak{p}$.

There is a structure theorem for local log-regular rings:

Theorem (Kato's structure theorem [Kato94])

$(R, \mathcal{Q}, \alpha)$: a local log ring satisfying the condition $(*)$.

k : the residue field of R, $r = \dim R/I_\alpha$.

Assume that $\mathcal{Q}$ is reduced.

1. Suppose that R is of equal characteristic.
   $(R, \mathcal{Q}, \alpha)$ is **a local log-regular ring** iff there exists

$$\begin{tikzcd} \mathcal{Q} \arrow[r] \arrow[d] & k[\![\mathcal{Q} \oplus \mathbb{N}^r]\!] \arrow[d, "\phi"] \\ R \arrow[r] & \widehat{R} \end{tikzcd}$$

   where ϕ is an **isomorphism**.

Theorem (continued)

2. Suppose that R is of mixed characteristic.
$(R, \mathcal{Q}, \alpha)$ is **a local log-regular ring** iff there exists

$$\begin{tikzcd} \mathcal{Q} \arrow[r] \arrow[d] & C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!] \arrow[d, "\phi"] \\ R \arrow[r] & \widehat{R} \end{tikzcd}$$

where

- $C(k)$ is a Complete DVR s.t. $C(k)/pC(k) \cong k$,
- ϕ is **surjective**,
- $\mathrm{Ker}(\phi)$ is generated by an element $\theta \in \mathfrak{m}_{C(k)[|\mathcal{Q} \oplus \mathbb{N}^r|]}$ whose constant term is p.

Theorem

$(R, \mathcal{Q}, \alpha)$: a local log-regular ring.
Then the following assertions hold:

1. R is Cohen-Macaulay and normal ([Kat94]).
2. A canonical module $\omega_R = \langle \alpha(q) \mid q \in \operatorname{relint} \mathcal{Q} \rangle$ ([Ish24]),
3. R has a pseudo-rational singularity ([Ish24]).

Several characterizations of local log-regular rings are also known.

Theorem (Characterizations of log-regularity)

$(R, \mathcal{Q}, \alpha)$: a local log ring satisfying the condition $(*)$.

Assume that R/I_α is regular.

The following assertions are equivalent:

1. $(R, \mathcal{Q}, \alpha)$ is a local log-regular ring,
2. For any $\mathfrak{q} \in \operatorname{Spec}(\mathcal{Q})$, $\mathfrak{q}R \in \operatorname{Spec}(R)$ such that $\alpha^{-1}(\mathfrak{q}R) = \mathfrak{q}$.

Other properties can be found in [Ogus], [Ish24], [Kat94], [CM22].

By the theorem of Chouinard (1981), we have

$$\mathrm{Cl}(A[\mathcal{Q}]) \cong \mathrm{Cl}(\mathcal{Q})$$

for any regular local ring A.
We proved an analogue of Chouinard's theorem for local log-regular rings.

Theorem ([GR], [Ish24])

$(R, \mathcal{Q}, \alpha)$: a local log-regular ring.
Then $\mathrm{Cl}(R) \cong \mathrm{Cl}(\mathcal{Q})$.

[Outline of the Proof of Theorem]

Step 1: We have the inclusions

$$\mathrm{Cl}(\mathcal{Q}) \hookrightarrow \mathrm{Cl}(R) \hookrightarrow \mathrm{Cl}(\widehat{R}).$$

Hence, we may assume that R is complete.

Step 2: We prove the following lemma.

Lemma

$(R, \mathcal{Q}, \alpha)$: a complete local log-regular ring. Then there exists a complete regular local ring T s.t.

- $R \hookrightarrow T$, and
- $S^{-1}R \cong S^{-1}T$ where the multiplicatively closed subset S is the image of $\mathcal{Q}$.

Step 3: We apply Nagata's theorem.

$$0 \to H \to \mathrm{Cl}(R) \to \mathrm{Cl}(S^{-1}R) \to 0$$

where $H = \langle [\mathfrak{p}] \in \mathrm{Cl}(R) \mid \mathfrak{p} \in \mathrm{Ht1}(R), \mathfrak{p} \cap S \neq \emptyset \rangle$.
By Step 2, we obtain $\mathrm{Cl}(S^{-1}R) \cong \mathrm{Cl}(S^{-1}T) = 0$.
$\rightsquigarrow H \cong \mathrm{Cl}(R)$.

Step 4: Connect $\mathrm{Cl}(\mathcal{Q})$ with H.

Lemma

$(R, \mathcal{Q}, \alpha)$: a local log-regular ring, $\mathfrak{p}$: a height one prime of R. Then the following assertions are equivalent.

1. There exists a height one prime ideal $\mathfrak{q}$ of $\mathcal{Q}$ such that $\mathfrak{p} = \mathfrak{q}R$,
2. The intersection of $\operatorname{Im}\alpha$ and $\mathfrak{p}$ is not empty.

We obtain

$$\mathrm{Cl}(\mathcal{Q}) \xrightarrow{\cong} \operatorname{Im}\big(\mathrm{Cl}(\alpha)\big) = \langle [\mathfrak{p}] \in \mathrm{Cl}(R) \mid \mathfrak{p} \in \mathrm{Ht1}(R), \mathfrak{p} \cap S \neq \emptyset \rangle = H.$$

where the first equality follows from the above Lemma.

In summary, we obtain $\mathrm{Cl}(\mathcal{Q}) \cong \mathrm{Cl}(R)$. $\square$

References

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