Gorenstein Dedekind Domains

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Presentation Overview

A Brief History of Gorenstein homological Algebra

- 1969: Auslander and Bridger develop the theory of stable module categories and introduce G-dimension[a].

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[c]Enochs, Edgar E.; Jenda, Overtoun M. G.: Gorenstein injective and projective modules, Math. Z. 220, No. 4, 611-633 (1995).

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- 2000s–present: Rapid development of Gorenstein homological dimensions, Gorenstein categories, and their applications.

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1. (WK) F. Wang and H. Kim, *Foundations of Commutative Rings and Their Modules*, 2nd ed., Springer, 2024.
2. (KMX) H. Kim, N. Mahdou, and S. Xing, *Gorenstein Homological Algebra*, in preparration.
3. (ZKH) D. Zhou, H. Kim, K. Hu, *Homological characterizations of G-Krull domains and G-Dedekind domains*, submitted.
4. (CHKQW) M. Chen, K. Hu, H. Kim, X. Qu, F. Wang, *A group action on Gorenstein projective modules and its application*, submitted.

Definition (Gorenstein projective module)

Let R be a ring. An R-module M is called Gorenstein projective (G-projective for short) if there exists an exact complex of projective R-modules

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to P^{-1} \to P^{-2} \to \cdots$$

such that $M = \ker(P_0 \to P^{-1})$ and such that $\mathrm{Hom}_R(\mathbf{P}, Q)$ is exact for every projective R-module Q.
The complex $\mathbf{P}$ is called a complete projective resolution of M.

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Definition (Gorenstein projective dimension)

Let R be a ring and M an R-module. The Gorenstein projective dimension of M, denoted by $\mathrm{G\text{-}pd}_R(M)$, is defined as the least non-negative integer n such that there exists an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$$

where each G_i is a Gorenstein projective R-module. If no such finite n exists, we say that $\mathrm{G\text{-}pd}_R(M) = \infty$.

Definition (Gorenstein injective module)

Let R be a ring. An R-module M is said to be Gorenstein injective (G-injective for short) if there exists an exact complex of injective R-modules

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Definition (Gorenstein injective dimension)

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$$0 \to M \to G^0 \to G^1 \to \cdots \to G^n \to 0$$

where each G^i is a Gorenstein injective R-module. If no such finite n exists, we say that $\text{G-id}_R(M) = \infty$.

Definition (Gorenstein global dimension)

The global Gorenstein dimension of a ring R, denoted by $\text{G-gl.dim}(R)$, is defined as

$$\begin{aligned} \text{G-gl.dim}(R) &:= \sup\{\text{G-pd}_R(M) \mid M \text{ is an } R\text{-module}\} \\ &= \sup\{\text{G-id}_R(M) \mid M \text{ is an } R\text{-module}\}. \end{aligned}$$

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Definition (Gorenstein flat modules)

Let R be a ring. An R-module M is called Gorenstein flat (G-flat for short) if there exists an exact complex of flat R-modules

$$\mathbf{F} = \cdots \to F_1 \to F_0 \to F^{-1} \to F^{-2} \to \cdots$$

such that $M = \ker(F_0 \to F^{-1})$ and for every injective right R-module I, the complex $I \otimes_R \mathbf{F}$ is exact.
The complex $\mathbf{F}$ is called a complete flat resolution of M.

Definition (Gorenstein hereditary rings and Gorenstein Dedekind domain)

[a] A ring R is called Gorenstein hereditary (G-hereditary for short) if every submodule of a projective module is Gorenstein projective (i.e., $\mathrm{G\text{-}gl.dim}(R) \leq 1$). If R is a G-hereditary domain, it is called Gorenstein Dedekind (G-Dedekind for short).

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1. *R is a G-hereditary ring.*
2. *Every factor module of a G-injective module is G-injective.*
3. *Every factor module of an injective module is G-injective.*

Definition

1. Let R be an integral domain with quotient field K. For $I \in \mathcal{F}(R)$, define $I^{-1} := \{x \in K \mid xI \subseteq R\}$ and $I_v := (I^{-1})^{-1}$. Then the map v from $\mathcal{F}(R)$ to $\mathcal{F}(R)$, given by $A \mapsto A_v$ for any $A \in \mathcal{F}(R)$, is a star operation, which is called the v-operation on R.

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4. A ring R is called Gorenstein semihereditary (G-semihereditary for short) if it is coherent and every submodule of a flat R-module is G-flat.
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6. A Noetherian domain is said to be a 1-Gorenstein domain if its self-injective dimension is at most 1.

Theorem

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8. *R is a 1-Gorenstein domain.*
9. *R is Noetherian and R_P is G-Dedekind for every $P \in \mathrm{Max}(R)$.*

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If R is a G-Dedekind domain, then $\dim(R) \leq 1$.

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If R is a G-Dedekind domain, then $\dim(R) \leq 1$.

Theorem

A domain is Dedekind if and only if it is integrally closed G-Dedekind.

Theorem (Hu, Wang, Xu, Zhao)

Let R be a one-dimensional Noetherian domain with quotient field K and integral closure $\overline{R}$. Then R is a G-Dedekind domain if and only if every prime ideal P of R containing $(R :_K \overline{R})$ is G-projective.

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Lemma (Hu and Wang)

Let R be a domain with $\mathrm{gl.\,dim}(R) = n < \infty$, and let $u \in R$ be a nonzero nonunit. Then

$$\text{G-gldim}(R/uR) \leq n - 1.$$

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Example (Hu, Wang, Xu, Zhao)

Let $D = \mathbb{Q}[y, z]$, where y and z are two indeterminates, and let $\mathbb{Q}$ denote the field of rational numbers. Then the ring

$$R = \mathbb{Q}[X^3, X^4] \cong D/(y^4 - z^3)$$

is a G-Dedekind domain. To verify this, observe that $\mathrm{gl.dim}(D) = 2$ and R is not a QF-ring. Then, by applying the above lemma, the result follows. However, the ring $\mathbb{Q} + X^3\mathbb{Q}[X]$ is an overring of R that is not G-Dedekind.

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2. An R-module M is called *n-copure projective* if $\operatorname{Ext}_R^1(M, N) = 0$ for every R-module $N \in \mathcal{F}_n$.
   A 0-copure projective module is simply called *copure projective*.
   The module M is said to be *strongly copure projective* if $\operatorname{Ext}_R^{i+1}(M, F) = 0$ for every flat R-module F and all $i \geq 0$.

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3. An R-module M is called *copure injective* if $\mathrm{Ext}^1_R(E, M) = 0$ for every injective R-module E.
4. A ring R is called a *CPH ring* if every submodule of a copure projective module is copure projective.

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   The module M is said to be strongly copure projective if $\mathrm{Ext}_R^{i+1}(M, F) = 0$ for every flat R-module F and all $i \geq 0$.
3. An R-module M is called copure injective if $\mathrm{Ext}_R^1(E, M) = 0$ for every injective R-module E.
4. A ring R is called a CPH ring if every submodule of a copure projective module is copure projective.
5. A module M over a domain R is said to be divisible if $rM = M$ for every nonzero $r \in R$.

Definition

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Theorem

[a] *The following statements are equivalent for a Noetherian domain R:*

1. *R is a CPH domain.*

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Theorem (Hu, Lim, Zhou)

[a] *An integral domain R is a G-Dedekind domain if and only if* $\mathrm{IPD}(R) \leq 1$.

[a] Hu, Kui; Lim, Jung Wook; Zhou, De Chuan, Flat dimensions of injective modules over domains, Bull. Korean Math. Soc. 57, No. 4, 1075-1081 (2020).

Definition (w-theory)

1. A nonzero ideal J of R is called a Glaz-Vasconcelos ideal (GV-ideal) if J is finitely generated and the natural homomorphism $\varphi : R \to \mathrm{Hom}_R(J, R)$ is an isomorphism.

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3. For a GV-torsion-free R-module M, set

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\},$$

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1. An ideal I of R is called a w-ideal (or semidivisorial ideal) if $I = I_w$.

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3. Let $\mathcal{F}(R)$ denote the set of nonzero fractional ideals of R. Then the map w from $\mathcal{F}(R)$ to $\mathcal{F}(R)$, given by $A \mapsto A_w$ for any $A \in \mathcal{F}(R)$, is a star operation, which is called the w-operation on R.

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4. Note that the w-operation is of finite character and stable.

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1. A Noetherian ring R is said to be Gorenstein if $\mathrm{id}_{R_{\mathfrak{m}}} R_{\mathfrak{m}} < \infty$ for any maximal ideal $\mathfrak{m}$ of R.
2. [a] A domain R is called a G-Krull domain if R satisfies the following three conditions:
   - (i) For each prime ideal $\mathfrak{p}$ of R of height one, $R_{\mathfrak{p}}$ is a Gorenstein ring.
   - (ii) $R = \bigcap R_{\mathfrak{p}}$, where $\mathfrak{p}$ ranges over all prime ideals of R of height one.
   - (iii) Any nonzero element of R lies in only a finite number of prime ideals of height one

[a] Qiao, L., Wang, F.G.: A half-centered star-operation on an integral domain. J. Korean Math. Soc. 54(1), 35–57 (2017).

Lemma

[a] *Let S be a multiplicative subset of R, M be an R-module, and N be an R_S-module. Then the natural R_S-homomorphism*

$$\theta : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_{R_S}(M_S, N)$$

is an isomorphism.

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Lemma (Characterization of SM domains)

A domain R is an SM domain if and only if $R_{\mathfrak{m}}$ is a Noetherian domain for any maximal w-ideal $\mathfrak{m}$ of R, and each nonzero element of R lies in only finitely many maximal w-ideals of R.

Theorem (Characterizations of G-Krull domains)

[a] *The following statements are equivalent for a domain R.*

1. *R is an SM domain and* $\mathrm{id}_R R_{\mathfrak{m}} \leq 1$ *for any maximal w-ideal* $\mathfrak{m}$ *of R.*

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5. *$R_{\mathfrak{m}}$ is a G-Dedekind domain for any maximal w-ideal $\mathfrak{m}$ of R, and each nonzero element of R lies in only finitely many maximal w-ideals of R.*
6. *R is a G-Krull domain.*

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2. Note that a domain R is a Krull domain if and only if R is a t-almost Dedekind domain and R is an SM domain.

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2. Note that a domain R is a Krull domain if and only if R is a t-almost Dedekind domain and R is an SM domain.

Example (w-locally G-Dedekind domain but not of w-finite character)

Let R be a non-Krull t-almost Dedekind domain[a]. Then, for any maximal w-ideal $\mathfrak{m}$ of R, $R_{\mathfrak{m}}$ is a discrete valuation ring, thus a G-Dedekind domain. However, R does not satisfy that each nonzero element of R lies in only finitely many maximal w-ideals of R. If not, we would get that R is an SM domain by Lemma 3. Thus, R would be a Krull domain, a contradiction.

[a] Kang, B.G.: Prüfer v-multiplication domains and the ring $R[X]_{N_v}$. J. Algebra 123, 151–170 (1989).

Lemma

[a] *Let* $(R, \mathfrak{m})$ *be a local Noetherian domain,* $\mathfrak{p}$ *a prime ideal of* R *with* $\mathfrak{p} \subsetneq \mathfrak{m}$*, and* M *a finitely generated* R*-module. If* $\mathrm{Ext}_R^{i+1}(R/Q, M) = 0$ *for any prime ideal* Q *properly containing* $\mathfrak{p}$*, then* $\mathrm{Ext}_R^i(R/\mathfrak{p}, M) = 0$*.*

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Theorem

Let $(R, \mathfrak{m})$ be a local Noetherian domain. If $\mathfrak{m}$ is a G-projective R-module, then R is a G-Dedekind domain.

Corollary

If R is a Noetherian domain and every maximal ideal of R is G-projective, then R is G-Dedekind.

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Definition

A GV-torsion-free R-module M is called a strong w-module if $\mathrm{Ext}^i_R(N, M) = 0$ for each $i \geq 1$ and any GV-torsion R-module N.

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Lemma

If $\mathrm{id}_R R \leq 1$, *then R is a strong w-module.*

Theorem (Characterizations of G-Dedekind domains)

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5. *R is an SM domain, K/R is a w-module, and $\mathrm{Ext}_R^1(I, R) = 0$ for any w-ideal I of R.*
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Theorem (Characterizations of G-Dedekind domains)

TFAE for a domain R with quotient field K.

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8. *R is a G-Dedekind domain.*

Question. The authors raised the question of whether every SG-Dedekind domain is necessarily a Dedekind domain. [a]

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Definition (strongly Gorenstein Dedekind domains)

1. An R-module M is called strongly Gorenstein projective (or SG-projective, for short) if and only if there exists a short exact sequence

$$0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0,$$

where P is a projective R-module, and $\mathrm{Hom}_R(-, Q)$ leaves the sequence exact for every projective R-module Q.
Let $\mathcal{SG}$ denote the class of SG-projective R-modules.

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   Let $\mathcal{SG}$ denote the class of SG-projective R-modules.
2. A domain R is called an SG-Dedekind domain if every submodule of any projective R-module is SG-projective.

Definition (Dimension related to $\mathcal{SG}$)

Let n be a non-negative integer and let M be an R-module.

1. We say that M has projective dimension with respect to $\mathcal{SG}$ (or $\mathcal{SG}$-projective dimension) at most n, denoted by $pd_{\mathcal{SG}}M \leq n$, if there exists a projective resolution

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

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2. The global dimension with respect to $\mathcal{SG}$ of R, denoted by $gl_{\mathcal{SG}}\dim(R)$, is defined as the supremum of the $\mathcal{SG}$-projective dimensions of all R-modules:

$$gl_{\mathcal{SG}}\dim(R) = \sup\{pd_{\mathcal{SG}}M \mid M \text{ is an } R\text{-module}\}.$$

Theorem (Characterizations of SG-Deekind domains)

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[a] (CHKQW)

Example

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Example

1. Let p be a prime number, and let $R := \mathbb{Z} + p\mathbb{Z}\mathrm{i}$. Then R is not a Dedekind domain. However, it is shown in [HKWXZ, Example 3.4] that every ideal of R is SG-projective.

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1. Let p be a prime number, and let $R := \mathbb{Z} + p\mathbb{Z}\mathrm{i}$. Then R is not a Dedekind domain. However, it is shown in [HKWXZ, Example 3.4] that every ideal of R is SG-projective.
2. Let p be a prime number, $R := \mathbb{Z} + p\mathbb{Z}\mathrm{i}$, and let $S := R_P$, where $P = (p, pi)$. It is shown in [HKWXZ, Example 3.5] that every ideal of S is SG-projective.

Open Questions

1. In an integral domain, the notions of an ideal being projective and being invertible are equivalent. Is there an invertibility property that is equivalent to being G-projective or SG-projective?
2. Dedekind domains are characterized by the fact that every nonzero ideal can be expressed as a product of prime ideals. Can G-Dedekind domains or SG-Dedekind domains be characterized in a similar way?
3. [a] Investigate if and how integral closure behaves differently for G-Dedekind domains compared to classical Dedekind domains.

[a] A. Geroldinger, H. Kim, A. Loper, On Long-Term Problems in Multiplicative Ideal Theory and Factorization Theory, to appear in Contemporary Mathematics Series of AMS

The End

Thank you for your attention.

Questions or Comments?