Modules whose endomorphism rings are reduced

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Overview

Introduction to endo-reduced modules

- R is a ring with unity, M is a unitary right R-module, and $S = \mathrm{End}_R(M)$ denote the endomorphism ring of M.
- R is **reduced** if it has no nonzero nilpotent elements.
- M_R is **reduced** if for each $m \in M$ and $a \in R$, $ma^2 = 0$ implies $mRa = 0$ (Lee & Zhou, 2004).
- M_R is **endo-reduced module** if S is a reduced ring.

Examples of endo-reduced modules

- Modules whose endomorphism rings are subdirect products of domains.
- Modules whose endomorphism rings are Abelian (von Neumann) regular
- $\mathbb{Q}_{\mathbb{Z}}, \mathbb{Z}_p, \mathbb{Z}_{\mathbb{Z}}, \mathbb{Z}_{p^\infty}, \mathbb{Q}/\mathbb{Z} = \bigoplus_{p \in \mathcal{P}} \mathbb{Z}_{p^\infty}$

Properties of endo-reduced modules

- ${}_SM$ is a reduced module $\Rightarrow$ M_R is an endo-reduced module
- M_R is an endo-reduced module $\not\Leftarrow$ M_R is a reduced module

- $\mathbb{Z}_n$ is a reduced module over $\mathbb{Z}$ iff n is square-free
- $\mathbb{Z}_{p^\infty}$ is endo-reduced but not reduced as S-module nor as a $\mathbb{Z}$-module

- A direct summand of an endo-reduced module is endo-reduced.
- A direct sum of endo-reduced modules is not endo-reduced.

- If $M = \bigoplus_{\alpha \in \mathcal{J}} M_\alpha$ is endo-reduced, then $\mathrm{Hom}(M_\alpha, M_\beta) = 0$ for all $\alpha \neq \beta \in \mathcal{J}$, Abelian, and Dedekind finite.

Properties of endo-reduced modules

Theorem

$M = \bigoplus_{\alpha\in\mathcal{J}} M_\alpha$ is endo-reduced if and only if M_α is endo-reduced for each $\alpha \in \mathcal{J}$ and $Hom(M_\alpha, M_\beta) = 0$ for all $\alpha \neq \beta \in \mathcal{J}$.

- Since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, Z_{q^\infty}) = 0$ for every $p \neq q$, we have $M_{\mathbb{Z}} = \mathbb{Q}/\mathbb{Z} = \bigoplus_{p\in\mathcal{P}} \mathbb{Z}_{p^\infty}$ endo-reduced

Corollary

If $\bigoplus_{\alpha\in\mathcal{J}\neq\emptyset} N_\alpha$ is endo-reduced with $N_\alpha = N$, then $|\mathcal{J}| = 1$

- The $\mathbb{Z}$-module $\mathbb{Z}[x](\cong \mathbb{Z}^{(\mathbb{N})})$ is not endo-reduced
- Simple modules $\{N_\alpha, \alpha \in \mathcal{J}\}$ (pairs-wise nonisomorphic), $\bigoplus_{\alpha\in\mathcal{J}} N_\alpha$ and $\prod_{\alpha\in\mathcal{J}} N_\alpha$ are endo-reduced

$\mathcal{K}$-nonsingular modules

- M is **nonsingular** if $\forall m \in M, mI = 0$ for $I \trianglelefteq_r R \Rightarrow m = 0$;
- M is $\mathcal{K}$-**nonsingular** if, $\forall \lambda \in \mathrm{End}(M), \ker(\lambda) \leqslant^{\mathrm{ess}} M$ implies $\lambda = 0$.
- A reduced ring is both left and right nonsingular, but this property need not hold for its modules.
- The $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is endo-reduced but not $\mathcal{K}$-nonsingular.

Theorem (Johnson and Wong, 1961)

If M is uniform and nonsingular, then $End_R(M)$ is a domain. If, in addition, M is quasi-injective, then $End_R(M)$ is a division ring.

Example

- The $\mathbb{Z}$-module $\mathbb{Z}_p$ is uniform, endo-reduced and $\mathcal{K}$-nonsingular but not nonsingular.

On $\mathcal{K}$-nonsingular modules

- For the broader class of uniform and $\mathcal{K}$-nonsingular modules, we have:

Theorem

If M is both uniform and $\mathcal{K}$-nonsingular, then M is an endo-reduced module. In particular, the following statements hold:

(1) every nonzero endomorphism of M is a monomorphism,

(2) $End_R(M)$ is a domain.

Furthermore, if M is quasi-injective, then $End_R(M)$ is a division ring.

Corollary

A $\mathcal{K}$-nonsingular uniform module is Hopfian (Every epimorphism is an isomorphism).

Over a commutative Dedekind domain

Proposition

Let R be a reduced ring and $M = \bigoplus_{\alpha \in \mathcal{J}} M_\alpha$ with cyclic R-modules M_α. If M is a torsion-free, then M is an endo-reduced module if and only if $M \cong R_R$.

- The cyclic torsion-free endo-reduced $\mathbb{Z}$-modules are precisely $\cong \mathbb{Z}$.

Theorem

Let $M = \bigoplus_{\alpha \in \mathcal{J}} M_\alpha$ be a direct sum of cyclic D-modules M_α over a commutative Dedekind domain D. Then the following statements are equivalent:

(1) *M is an endo-reduced module,*

(2) *M is either a semisimple module with pair-wise non-isomorphic submodules or M is a torsion-free module which is isomorphic to D_D.*

Over a commutative Dedekind domain

Example

A finitely generated Abelian group G is endo-reduced as a $\mathbb{Z}$-module if and only if one of the following two conditions holds:

(1) $G \cong \bigoplus_{p_\alpha \in \mathcal{P}} \mathbb{Z}_{p_\alpha}$, where $\mathcal{P}$ is a collection of distinct prime numbers p_α of $\mathbb{Z}$ and $\alpha \in \mathbb{Z}^+$;

(2) $G \cong \mathbb{Z}_\mathbb{Z}$.

- The "finitely generated" hypothesis is not superfluous.

Example

The $\mathbb{Z}$-module $\mathbb{Q}$ is a non-finitely generated (Abelian group), endo-reduced but fails on the (1) and (2) above.

Bibliography

- Johnson, R. E., & Wong, E. T. (1961). Quasi-injective modules and irreducible rings. *J. Lond. Math. Soc.*, **1**, 260–268.
- Kimuli, P. I., & Ssevviiri, D. (2024). Modules with reduced endomorphism rings. *J. Algebra Appl.*,2650042.
- Lee, T. K., & Zhou, Y. (2004). *Reduced Modules. Rings, Modules, Algebras and Abelian Group. Lecture Notes in Pure and Applied Mathematics.*. Marcel Dekker, New York, 236, 365–377
- Rizvi, S. T., & Roman, C. S. (2007). On $\mathcal{K}$-nonsingular modules and applications. *Comm. Algebra* 35(9): 2960–2982.

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