Waring problem for real polynomial rings

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Rings and Polynomials

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If $S \subset R$ is a multiplicative set then $w_n(R) \geq w_n(S^{-1}R)$. If $\varphi : R \to S$ is an epimorphism then $w_n(R) \geq w_n(S)$.

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If n is odd then $w_n(\mathbb{Z}) \leq g(n)$.

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If $s_n(R)$ is finite then there is a similar formula

$$n!x = \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{n-1}{r} [(x+r)^n - r^n]$$

which implies (when $n!$ is a unit) $w_n(R) \leq nG(n)(s_n(R) + 1)$.

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- in particular $w_2(\mathbb{R}(x, y)) = 4$,
- for any n we have $w_{2n}(\mathbb{Z}[x]) = \infty$ - Choi, Dai, Lam, Reznick,
- $3 \leq w_4(\mathbb{R}(x)) \leq 6$ - Choi, Lam, Prestel, Reznick

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We define the family of polynomials:

- $F_1 = 1$,
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F_n *needs* n *squares and not less.* □

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Corollary

Let $(R, \mathfrak{m})$ *be a regular local ring of dimension at least 3 and with real residue field. Then* $w_{2d}(R) = \infty$ *for any* $d \geq 1$.

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If F is a field of finite transcendence degree n over $\mathbb{R}$ then $w_2(F) \leq 2^n$

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No similar theorem for general rings are known.
Proof of the first part relies on the theory of multiplicative quadratic forms developed by Pfister.

Definition

We define the ring of regular functions $\mathcal{O}(\mathbb{R}^2)$ on $\mathbb{R}^2$ as

$$\mathcal{O}(\mathbb{R}^2) = \left\{ \frac{f}{g} \mid f, g, \in \mathbb{R}[x, y],\ g^{-1}(0) = \varnothing \right\}.$$

A bit of real algebraic geometry

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$$\mathcal{O}(\mathbb{R}^2) = \left\{ \frac{f}{g} \mid f, g, \in \mathbb{R}[x, y],\ g^{-1}(0) = \varnothing \right\}.$$

Obviously $w_2(\mathcal{O}(\mathbb{R}^2)) \geq w_2(\mathbb{R}(x, y)) = 4$.

Theorem

Let K be a field of characteristic different from 2. Let φ be a quadratic forms with coefficients in K and $f \in K[x]$ be a polynomial. If f is represented by φ over $K(x)$ then it is represented by φ already over $K[x]$.

Back to real algebraic geometry

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Sketch of a proof: Let $f \in \mathcal{O}(\mathbb{R}^2)$ be a sum of squares. We may assume that the zero set of f is finite and f is a polynomial. Of course, $f \in \mathbb{R}[x, y] \subset \mathbb{R}(x)[y] \subset \mathbb{R}(x, y)$.

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Bruce Reznick posed the following:

Conjecture

Let $f \in \mathbb{R}[x, y]$ be a homogeneous polynomial of degree $4d$ which is a sum of fourth powers of polynomials. Then we can write $f = f_1^2 + f_2^2$ where f_1, f_2 are nonnegative.

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This conjecture would imply $w_4(\mathbb{R}[x]) \leq 6$ because of the formula

$$(p_1^2 + q_1^2)^2 = \frac{1}{18}((\sqrt{3}p_1 + q_1)^4 + (\sqrt{3}p_1 - q_1)^4 + 16q_1^4).$$

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Sadly, this conjecture is false [K. Vill 2023]. There is a lot (an open subset) of polynomials which cannot be "doubly positive".

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4. Compute $w_{2d}(\mathrm{Int}(\mathbb{Z}))$, where $\mathrm{Int}(\mathbb{Z})$ is the ring of integer valued polynomials. This is not known even for $d = 1$.

Thank you! :)