Generalised Reduced Modules

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Throughout the talk, R denotes a commutative unital ring.

Let R be a ring and M be an R-module.

- R is reduced if for all $a \in R$, $a^2 = 0$ implies that $a = 0$.
- M is reduced if for all $a \in R$ and $m \in M$, $a^2m = 0$ implies that $am = 0$.
- Reduced modules were later generalised to a-reduced modules for $a \in R$.

Definition 1.1 (A. Kyomuhangi, D. Ssevviiri, 2020)

An R-module M is a-reduced if for all $m \in M$, $a^2m = 0$ implies that $am = 0$. As such, M is reduced if it is a-reduced for all $a \in R$.

We now generalise the notion of a-reduced and reduced.

Definition 2.1 (A. Kyomuhangi, D. Ssevviiri, 2023)

Let $a \in R$ and $t \in \mathbb{Z}^+$ be fixed. An R-module M is

- *a^t-reduced if for all $m \in M$ and all positive integers $k \geq t$, $a^k m = 0$ implies that $a^t m = 0$.*
- *universally a^t-reduced (written as ε^t-reduced) if M is a^t-reduced for every $a \in R$.*
- *$t = 1$ retrieves a-reduced and reduced modules.*

Examples

- Finitely generated modules over a Noetherian ring,
- a flat (resp. free, projective) module over an a^t-reduced ring R.

Proposition 2.2

For $t \in \mathbb{Z}^+$ and $a \in R$, we have

$$\begin{tikzcd}
 & \boxed{a\text{-reduced}} \arrow[dr] & \\
\boxed{\text{Reduced}} \arrow[ur] \arrow[dr] & & \boxed{a^t\text{-reduced}} \\
 & \boxed{\varepsilon^t\text{-reduced}} \arrow[ur] &
\end{tikzcd}$$

Examples

- The $\mathbb{Z}$-module $\mathbb{Z}_{2^3}$ is 2^3-reduced but not 2-reduced, (a^t-reduced $\not\Rightarrow$ a-reduced)
- The $\mathbb{Z}$-module $\mathbb{Z}_{16}$ is 4^2-reduced but not 2^2-reduced. (a^t-reduced $\not\Rightarrow$ ε^t-reduced)
- The $\mathbb{Z}$-module $\mathbb{Z}_4$ is 4-reduced but not 2-reduced(a-reduced $\not\Rightarrow$ reduced)
- If $R := \mathbb{Z}_8$, $M := \mathbb{Z}_4$ and $t := 2 \in \mathbb{Z}^+$, then $\mathbb{Z}_4$ is ε^2-reduced but not reduced.

Definition 3.1

A ring R is (von Neumann) regular if for every $a \in R$, there exists $b \in R$ such that $a = a^2 b$.

Definition 3.2 (N. H. McCoy, 1939)

A ring R is t-regular if for every $a \in R$, a^t is regular; i.e., for every $a \in R$, there exists $b \in R$ such that $a^t = a^{2t} b$.

- A regular ring is reduced. For t-regular rings, we have:

Proposition 3.3

If R is a t-regular ring such that every ideal of R of the form $(0 : b)$ for some $0 \neq b \in R$ is semiprime, then R is ε^t-reduced.

Corollary 3.4

A Noetherian t-regular ring is ε^t-reduced.

Theorem 3.5

Let every ideal of R of the form $(0 : b)$ for some $0 \neq b \in R$ be semiprime. TFAE:

1. *every R-module is ε^t-reduced,*
2. *every cyclic R-module is ε^t-reduced,*
3. *R is t-regular.*

Proposition 3.6

Let R be a ring such that for every R-module M and $0 \neq m \in M$, $(0 : m)$ is a semiprime ideal of R. M is reduced if and only if it is ε^t-reduced.

A functor $\gamma : R\text{-Mod} \to R\text{-Mod}$ is

- a preradical if for every $f \in \mathrm{Hom}_R(M, N)$, $f(\gamma(M)) \subseteq \gamma(N)$.
- a radical if γ is a preradical and $\gamma(M/\gamma(M)) = 0$
- left exact if for any exact sequence $0 \to N \to M \to K$ of R-modules, the sequence $0 \to \gamma(N) \to \gamma(M) \to \gamma(K)$ is exact.

Let I be an ideal in R, $M \in R$-Mod.

- An $m \in M$ is an I-torsion of M if $I^k m = 0$ for some $k \in \mathbb{Z}^+$.

Taking $\Gamma_I(M)$ to be the set of all I-torsion elements in M, i.e., $\Gamma_I(M) := \left\{m \in M \mid I^k m = 0 \text{ for some } k \in \mathbb{Z}^+\right\}$,

- $\Gamma_I(M) \leq M$
- $\Gamma_I(M) \cong \varinjlim_k \operatorname{Hom}_R(R/I^k, M)$.

Definition 4.1 (I-torsion functor)

The I-torsion functor is $\Gamma_I : R\text{-}Mod \to R\text{-}Mod,\ M \mapsto \Gamma_I(M)$.

- Γ_I is left exact and for $I = (a)$, $\Gamma_I(M) := \Gamma_a(M)$.

- If R is a Noetherian ring, then Γ_I
  - is a left exact radical on R-Mod.
  - preserves injective modules.

The i-th right derived functor is the i-th local cohomology functor with respect to I given by

$$H_I^i(-) \cong \varinjlim_k \operatorname{Ext}_R^i(R/I^k, -) \text{ for each } i \geq 0$$

Proposition 4.2

Let $M \in R$-Mod and $I = (a)$, where $a \in R$. TFAE:

- *M is a^t-reduced;*
- $a^t\Gamma_a(M) = 0$;
- $(0 :_M a^k) = (0 :_M a^t)$ *for all* $k \geq t \in \mathbb{Z}^+$;
- $Hom_R(R/I^k, M) \cong Hom_R(R/I^t, M)$ *for all* $k \geq t \in \mathbb{Z}^+$;
- $\Gamma_a(M) \cong Hom_R(R/I^t, M)$.

For each $a \in R$ and $t \in \mathbb{Z}^+$, we define the functor $a^t\Gamma_a : R\text{-Mod} \to R\text{-Mod},\ M \mapsto a^t\Gamma_a(M)$ given by $a^t\Gamma_a(M) := \left\{a^t m \mid a^k m = 0,\ m \in M, \text{for some } k \geq t \in \mathbb{Z}^+\right\}$.

- $a^t\Gamma_a$ is a radical, called the generalised locally nilradical
- $a^t\Gamma_a$ in general is not left exact.
- An R-module M is a^t-reduced if $a^t\Gamma_a(M) = 0$ and ε^t-reduced if $a^t\Gamma_a(M) = 0$ for all $a \in R$.

Thus, $a^t\Gamma_a$ measures how far a module is from being a^t-reduced.

Proposition 5.1

For any positive integer t and $a \in R$, $a^t\Gamma_a(R[x]) = a^t\Gamma_a(R)[x]$

Corollary 5.2

$R[x]$ is a^t-reduced if and only if R is a^t-reduced.

Proposition 5.3

If R is a Noetherian ring, $a \in R$, I the ideal of R generated by a and M an a^t-reduced R-module, then the i-th local cohomology module $H^i_I(M)$ is given by

$$H^i_I(M) \cong Ext^i_R(R/I^t, M)$$

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Thank you for your attention!