On lower bounds for the number of conjugacy classes of a finite group

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A theorem of Landau

Let G be a finite group. Let $k(G)$ be the number of conjugacy classes of G.

Answering a question of Frobenius, in 1903 Landau proved the following. For every positive integer k there are at most finitely many finite groups G with $k(G) = k$.

Landau's proof gives

$$k(G) \geq c \log_2 \log_2 |G|$$

for some constant c with $0 < c < 1$ (and $|G| > 1$).

Brauer's 3rd problem

Brauer's 3rd problem (1963).

Let G be a finite group. Give an asymptotically better lower bound for $k(G)$ only in terms of $|G|$ than the one which follows from Landau's theorem.

The solution for Brauer's problem

Theorem (Pyber (1992)).

There exists $\epsilon > 0$ such that for all finite groups G of order at least 4, we have

$$k(G) > \epsilon \cdot \frac{\log_2 |G|}{(\log_2 \log_2 |G|)^8}.$$

Keller (2011) showed that the 8 can be replaced by 7.

Baumeister, M, Tong-Viet (2016) proved that the 7 could be replaced by $3 + \delta$.

Question (Bertram).

Is it true that for any finite group G we have $k(G) \geq \log_3 |G|$?

More on Landau's theorem, Part I

Let $\mathrm{kpp}(G)$ denote the number of conjugacy classes of the finite group G consisting of elements of prime power orders.

Héthelyi and Külshammer (2005) proved that there exists a function f on the set of natural numbers such that

$$\mathrm{kpp}(G) \geq f(|G|)$$

for all finite groups G and $f(x) \to \infty$ as $x \to \infty$.

More on Landau's theorem, Part II

For a prime p, let $k_p(G)$ denote the number of conjugacy classes of non-trivial p-elements in G.

Theorem (Çınarcı, Keller, M, Simion (2025)).

There exists a function $f(x)$ with $f(x) \to \infty$ as $x \to \infty$ such that

$$\max_p \{k_p(G)\} \geq f(|G|)$$

for any finite group G.

This theorem was possible to prove using a very recent result of Giudici, Morgan and Praeger which is a slightly stronger version of the theorem for G a nonabelian finite simple group.

More on Landau's theorem, Part III

Landau's theorem does not depend on the Classification of Finite Simple Groups. Both the Héthelyi-Külshammer and the Giudici-Morgan-Praeger theorems do. So does ours.

In our theorem the function $f(x)$ is unspecified. The reason is that the function in the Giudici-Morgan-Praeger theorem is unspecified.

If we restrict our attention to solvable groups G, then (in our proof) the corresponding function $f(x)$ grows slower than 23 iterated logarithms (with base 2).

Lower bounds (for the number of conjugacy classes) in terms of a prime

Theorem (Héthelyi, Külshammer (2000)).

If G is a finite solvable group and p a prime divisor of the order of G, then

$$k(G) \geq 2\sqrt{p-1}.$$

Theorem (M (2016)).

If G is a finite group and p a prime divisor of the order of G, then

$$k(G) \geq 2\sqrt{p-1}$$

with equality if and only if $\sqrt{p-1}$ is an integer $G = C_p \rtimes C_{\sqrt{p-1}}$ and $C_G(C_p) = C_p$.

Lower bounds (for the number of characters) in terms of a prime

Theorem (Malle, M (2016)).

Let G be a finite group and p a prime divisor of the order of G. Then

$$|\mathrm{Irr}_{p'}(G)| \geq 2\sqrt{p-1}.$$

Theorem (Hung, Schaeffer Fry (2022)).

If B is the principal p-block of a finite group G where p is a prime divisor of the order of G, then

$$k(B) \geq 2\sqrt{p-1}.$$

Special conjugacy classes

For a prime p and a finite group G, let $k_p(G)$ denote the number of conjugacy classes of nontrivial p-elements in G and let $k_{p'}(G)$ be the number of conjugacy classes of p'-elements in G.

Theorem (Hung, M (2022)).

If p divides the order of a finite group G, then

$$k_p(G) + k_{p'}(G) \geq 2\sqrt{p-1}$$

with equality if and only if $\sqrt{p-1}$ is an integer and $G = C_p \rtimes C_{\sqrt{p-1}}$ is a Frobenius group (when $p > 2$) or $G = C_2$ (when $p = 2$).

Our second theorem

Theorem (Çınarcı, Keller, M, Simion (2025)).

Let G be a finite group and p a prime dividing $|G|$. One of the following holds.

(i) There exists a factorization $p-1=ab$ with a and b positive integers such that $k_p(G) \geq a$ and $k_{p'}(G) \geq b$, with equality in both cases if and only if $G = C_p \rtimes C_b$ such that $C_G(C_p) = C_p$.

(ii) $p = 11$ and $G = C_{11}^2 \rtimes \mathrm{SL}(2,5)$.

The inequalities $k_p(G) \geq a$ and $k_{p'}(G) \geq b$ follow immediately from Brauer's work in the case that G has a Sylow p-subgroup of order p. The theorem was also known for all groups G with $k_p(G) \leq 3$ by work of Hung, Sambale, Tiep (2024).

Structure of the proof, Part I

1. Groups with cyclic Sylow p-subgroups P. There is an elementary reduction to the case when P has order p. Brauer's work is used, noting that $k_{p'}(G)$ is the number of irreducible Brauer characters in G.
2. Groups which are not p-solvable. Earlier papers on simple groups are used.
3. Reduction to the case when $G = HV$ where V is an irreducible and faithful H-module for a finite group H of order coprime to $|V|$ (a power of p).
4. Primes p at most 43.
5. The case where H is an almost quasisimple group.
6. The case where H is a metacyclic group.
7. Counting orbits of H on V. Structure theorems.
8. Using the structure theorems to deal with primes p of medium size.

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