Ideal Factorization in Leavitt Path Algebras

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Directed Graphs

- A (*directed*) *graph* $E = (E^0, E^1, \mathbf{s}, \mathbf{r})$ consists of two sets E^0, E^1 (the elements of which are called *vertices* and *edges*, respectively), together with functions $\mathbf{s}, \mathbf{r} : E^1 \to E^0$, called *source* and *range*, respectively.

- A vertex $v \in E^0$ for which $\{e \in E^1 \mid \mathbf{s}(e) = v\}$ is finite and nonempty is called *regular*.

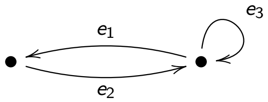
- A *path* μ in E is a finite sequence of edges $\mu = e_1 \cdots e_n$ such that $\mathbf{r}(e_i) = \mathbf{s}(e_{i+1})$ for $i = 1, \ldots, n-1$. We define $\mathbf{s}(\mu) := \mathbf{s}(e_1)$ to be the *source* of μ, and $\mathbf{r}(\mu) := \mathbf{r}(e_n)$ to be the *range* of p.

- A path $\mu = e_1 \cdots e_n$ in E is a *cycle* if $\mathbf{s}(\mu) = \mathbf{r}(\mu)$ and $\mathbf{s}(e_i) \neq \mathbf{s}(e_j)$ for all $i \neq j$. An *exit* for $\mu = e_1 \cdots e_n$ is an edge $f \in E^1 \setminus \{e_1, \ldots, e_n\}$ that satisfies $\mathbf{s}(f) = \mathbf{s}(e_i)$ for some i.

- The *extended graph* F of E has $F^0 = E^0$ and $F^1 = E^1 \cup \{e^* \mid e \in E^1\}$, where $\mathbf{s}, \mathbf{r} : F^1 \to F^0$ agree with the source and range maps of E on elements of E^1, and $\mathbf{s}(e^*) = \mathbf{r}(e)$, $\mathbf{r}(e^*) = \mathbf{s}(e)$ for all $e \in E^1$.

Extended Graphs

Let E be the following graph.



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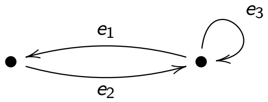
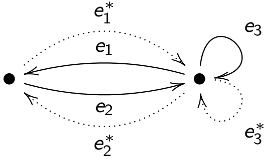


Then the extended graph F of E is as follows.



Leavitt Path Algebras

- From now on, K will denote an arbitrary field.

Definition

Given a graph $E = (E^0, E^1, \mathbf{s}, \mathbf{r})$, the *Leavitt path K-algebra* (*LPA*) $L_K(E)$ *of* E is the K-algebra generated by $\{v \mid v \in E^0\} \cup \{e, e^* \mid e \in E^1\}$, subject to the relations:

(V) $vw = \delta_{v,w} v$ for all $v, w \in E^0$,
(E1) $\mathbf{s}(e)e = e\mathbf{r}(e) = e$ for all $e \in E^1$,
(E2) $\mathbf{r}(e)e^* = e^*\mathbf{s}(e) = e^*$ for all $e \in E^1$,
(CK1) $e^*f = \delta_{e,f}\mathbf{r}(e)$ for all $e, f \in E^1$,
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- $L_K(E)$ is unital if and only if E^0 is finite.

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- Leavitt path algebras were defined independently by Abrams/Aranda Pino (2005) and Ara/Moreno/Pardo (2007).

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- $L_K(E)$ is unital if and only if E^0 is finite.
- Leavitt path algebras were defined independently by Abrams/Aranda Pino (2005) and Ara/Moreno/Pardo (2007).
- Leavitt path algebras are algebraic analogues of the graph C^*-algebras.

Examples of LPAs

- For any $n \geq 1$, $\mathbb{M}_n(K) \cong L_K(E)$, where E is the following graph.

$$\bullet^{v_1} \longrightarrow \bullet^{v_2} \longrightarrow \bullet^{v_3} \cdots\cdots \bullet^{v_{n-1}} \longrightarrow \bullet^{v_n}$$

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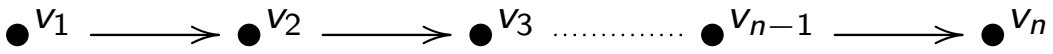

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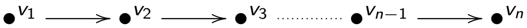

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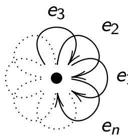
- For any $n \geq 2$, $L_K(n) \cong L_K(E)$, where E is the following graph.



The Leavitt K-algebra $L_K(n)$ is universal for the property that $L_K(n) \cong L_K(n)^n$ as right $L_K(n)$-modules, but $L_K(n) \not\cong L_K(n)^m$ for all $2 \leq m < n$.

Ideal Lattices

- A partially-ordered set $(L, \leq)$ is a *lattice*, if for all $a, b \in L$ there exists in L an infimum (*meet*) $a \wedge b$ and a supremum $a \vee b$ (*join*).
- A lattice L is *complete* if every $M \subseteq L$ has a meet $\bigwedge M$ and a join $\bigvee M$.
- Let L be a complete lattice, and $a \in L$. Then a is *compact* if for all $M \subseteq L$ such that $a \leq \bigvee M$, there exist $b_1, \ldots, b_n \in M$ satisfying $a \leq (b_1 \vee \cdots \vee b_n)$.
- A complete lattice L is *algebraic* if every element of L is the join of a set of compact elements.

Fact

For any ring R, the set $\mathcal{I}(R)$ of ideals of R forms an algebraic lattice, with set intersection as meet and ideal addition as join.

Ideal Lattices of LPAs

- A lattice L is *distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in L$.
- Let L be a complete lattice, and $a \in L$. Then a is *supercompact* if for all $M \subseteq L$ such that $a \leq \bigvee M$, there exists $b \in M$ satisfying $a \leq b$.
- A complete lattice L is *superalgebraic* if every element of L is the join of a set of supercompact elements.

Theorem (Miller, 2025)

For every distributive superalgebraic lattice L there exists a graph E such that

$$L \cong \mathcal{I}(L_K(E)).$$

Unreasonable Commutativity of Ideals in LPAs

Fact

Let R be any commutative ring. Then $IJ = JI$ for all ideals I and J in R.

Theorem (Rangaswamy, 2017)

Let E be any graph. Then $IJ = JI$ for all ideals I and J in $L_K(E)$.

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Definition

An integral domain in which every finitely generated ideal is principal is called a *Bézout* domain.

Theorem (Rangaswamy, 2014)

Every finitely generated ideal of $L_K(E)$ is principal, for any graph E.

Unreasonable Commutativity of Ideals in LPAs (Continued)

Definition

A ring R is *arithmetical* if the ideal lattice $\mathcal{I}(R)$ is distributive, that is

$$I_1 \cap (I_2 + I_3) = I_1 \cap I_2 + I_1 \cap I_3$$

for all ideals I_1, I_2, I_3 of R.

An arithmetical integral domain is called a *Prüfer* domain.

Theorem (Rangaswamy, 2017)

$L_K(E)$ is arithmetical, for any graph E.

Unreasonable Commutativity of Ideals in LPAs (Continued)

Definition

A *Dedekind domain* is an integral domain in which every proper ideal factors into a (finite) product of prime ideals.

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Which ideals in $L_K(E)$ can be factored into products of prime ideals?

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For which graphs E is it the case that every ideal of $L_K(E)$ can be factored into a product of prime ideals?

More Graph Terminology

- Let $E = (E^0, E^1, \mathbf{s}, \mathbf{r})$ be a graph.
- If $u, v \in E^0$ and there is a path μ in E satisfying $\mathbf{s}(\mu) = u$ and $\mathbf{r}(\mu) = v$, then we write $u \geq v$.
- $H \subseteq E^0$ is *hereditary* if whenever $u \in H$ and $u \geq v$ for some $v \in E^0$, then $v \in H$.
- $H \subseteq E^0$ is *saturated* if $\mathbf{r}(\mathbf{s}^{-1}(v)) \subseteq H$ implies that $v \in H$, for any regular $v \in E^0$.
- A nonempty $M \subseteq E^0$ is a *maximal tail* if $E^0 \setminus M$ is hereditary and saturated, and M is downward directed (i.e., for all $u, v \in M$, there exists $w \in M$ such that $u \geq w$ and $v \geq w$.).
- A *breaking vertex* of a hereditary saturated $H \subseteq E^0$ is an infinite emitter $w \in E^0 \setminus H$ such that $0 < |\mathbf{s}^{-1}(w) \cap \mathbf{r}^{-1}(E^0 \setminus H)| < \aleph_0$. The set of all breaking vertices of H is denoted by B_H.
- Given a hereditary saturated $H \subseteq E^0$ and $S \subseteq B_H$, (H, S) is called an *admissible pair*.

Ideals in LPAs

Definition

Let E be a graph. For an admissible pair (H, S) in E, define the *quotient graph* $E\backslash(H, S)$ via $(E\backslash(H, S))^0 = (E^0\backslash H) \cup \{v' \mid v \in B_H\backslash S\}$,

$$(E\backslash(H, S))^1 = \{e \in E^1 \mid r(e) \notin H\} \cup \{e' \mid e \in E^1 \text{ with } \mathbf{r}(e) \in B_H\backslash S\},$$

and extending $\mathbf{r}, \mathbf{s}$ to $E\backslash(H, S)$ by setting $\mathbf{s}(e') = \mathbf{s}(e)$ and $\mathbf{r}(e') = \mathbf{r}(e)'$.

Theorem (Rangaswamy, 2014)

Let I be an ideal of $L_K(E)$, with $H = I \cap E^0$ and $S = \{v \in B_H \mid v^H \in I\}$, where $v^H = v - \sum_{\mathbf{s}(e)=v,\, \mathbf{r}(e)\notin H} ee^*$. Then

$$I = I(H, S) + \sum_{i \in Y} \langle f_i(c_i) \rangle,$$

where $I(H, S)$ the ideal generated by $H \cup \{v^H \mid v \in S\}$, Y is an index set, each c_i is a cycle without exits in $E\backslash(H, S)$, and each $f_i(x) \in K[x] \setminus \langle x \rangle$.

Prime Factorizations in LPAs

Theorem (2020)

The following are equivalent for any graph E and proper ideal I of $L_K(E)$.

1. I is a product of prime ideals.
2. $I = I(H,S) + \sum_{i=1}^{k} \langle f_i(c_i) \rangle$, where each c_i is a cycle without exits in $E \setminus (H,S)$; each $f_i(x) \in K[x] \setminus \langle x \rangle$; $(E \setminus (H,S))^0$ is the union of $n \in \mathbb{Z}^+$ maximal tails; and $0 \leq k \leq n$.

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Theorem (2020)

The following are equivalent for any graph E.

1. Every proper ideal of $L_K(E)$ is a product of prime ideals.
2. There are only finitely many prime ideals minimal over any proper non-prime ideal of $L_K(E)$.
3. For every admissible pair (H,S) with $H \neq E^0$, $(E \setminus (H,S))^0$ is the union of $n \in \mathbb{Z}^+$ maximal tails, and there are at most n cycles without exits in $E \setminus (H,S)$.

Semiprime Factorizations in LPAs

Theorem (2020)

The following are equivalent for any graph E and proper ideal I of $L_K(E)$.

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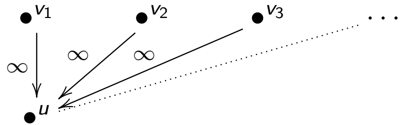
The following are equivalent for any graph E.

1. Every proper ideal of $L_K(E)$ is a product of semiprime ideals.
2. For any hereditary saturated $H \subseteq E^0$, there are only finitely many cycles c in E whose vertices do not lie in H, but $\mathbf{r}(e) \in H$ for all exits e of c.

An Example

- Let E be the following (infinite clock) graph.



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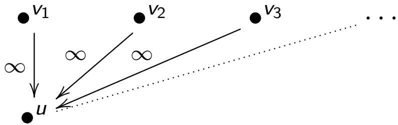

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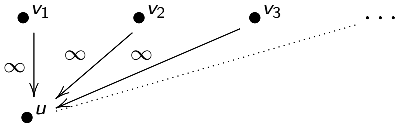

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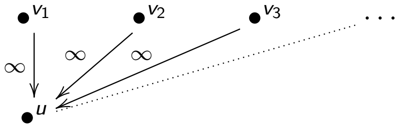

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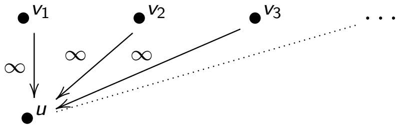

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- Every proper ideal of $L_K(E)$ is a product of semiprime ideals, since E is acyclic.
- $(\{u\}, \emptyset)$ is an admissible pair in E, with $E \setminus (\{u\}, \emptyset)$ as follows.

- $E \setminus (\{u\}, \emptyset)$ is not the union of finitely many maximal tails, and hence $I(\{u\}, \emptyset)$ is a proper ideal of $L_K(E)$ that is not a product of prime ideals.

Prime-Like Ideals

Definition

Let R be a ring, and let I be a proper ideal of R.

1. The *radical* of I, denoted $\mathrm{rad}(I)$ (or $\sqrt{I}$), is the intersection of all the prime ideals of R containing I.
2. I is *primary* in case for all ideals A and B of R, $B \subseteq \mathrm{rad}(I)$ whenever $AB \subseteq I$ and $A \not\subseteq I$.
3. I is *quasi-primary* in case $\mathrm{rad}(I)$ is prime.
4. I is *irreducible* in case for all ideals A and B of R, $I = A \cap B$ implies that $I = A$ or $I = B$.
5. I is *prime-power* in case $I = P^n$ for some prime ideal P of R and $n \in \mathbb{Z}^+$.

Prime-Like Ideals in LPAs

Proposition (2020)

The following are equivalent for any graph E and proper ideal I of $L_K(E)$.

1. I is primary.
2. I is quasi-primary.
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Corollary

The following are equivalent for any graph E and proper ideal I of $L_K(E)$.

1. I is a product of prime ideals.
2. I is a product of primary ideals.
3. I is a product of quasi-primary ideals.
4. I is a product of irreducible ideals.
5. I is a product of prime-power ideals.

More Prime-Like Ideals

Definition

Let R be a ring, and let I be a proper ideal of R.

1. I is *completely irreducible* if I is not the intersection of any set of ideals properly containing I.
2. I is *strongly prime* if for all ideals J_i $(i \in Y)$ of R, $\bigcap_{i \in Y} J_i \subseteq I$ implies that $J_i \subseteq I$ for some $i \in Y$.

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Definition

Let E be a graph, and let $S \subseteq E^0$ be nonempty.

1. S satisfies the *countable separation property* (CSP) if there is a countable $T \subseteq S$, such that for every $u \in S$ there is a $v \in T$ satisfying $u \geq v$.
2. S satisfies the *strong CSP* if S satisfies the CSP with respect to some countable $T \subseteq S$, such that T is contained in every nonempty hereditary saturated subset of S.

Completely Irreducible Ideals in LPAs

Theorem (2022)

The following are equivalent for any graph E and proper ideal I of $L_K(E)$.

1. I is the product of completely irreducible ideals.
2. $I = I(H,S) + \sum_{i=1}^{n} \langle f_i(c_i) \rangle$, where
   - $0 \leq n$ is an integer, with $n = 0$ indicating that $I = I(H,S)$;
   - each c_i is a cycle without exits in $E \setminus (H,S)$, and each $f_i(x) \in K[x] \setminus \langle x \rangle$;
   - $(E \setminus (H,S))^0 = \bigcup_{i=1}^{m} M_i$, where each M_i is a maximal tail, $n \leq m$, for each $i \leq n$ we have $s(c_i) \in M_i$ and $s(c_i) \notin M_j$ for $j \neq i$, and for $n+1 \leq i \leq m$ every cycle with source in M_i has an exit and M_i satisfies strong CSP.

Theorem (2022)

The following are equivalent for any graph E.

1. Every proper ideal of L is the product of completely irreducible ideals.
2. E satisfies Condition (K) (every cycle source is the source of multiple cycles), and for each admissible pair (H,S) with $H \neq E^0$, $(E \setminus (H,S))^0$ is the union of finitely many maximal tails satisfying strong CSP.

Strongly Prime Ideals in LPAs

Theorem (Aljohani/Radler/Rangaswamy/Srivastava, 2021)

The following are equivalent for any graph E and proper ideal I of $L_K(E)$.

1. I is a product of strongly prime ideals.
2. $I = I(H, S)$ and $(E \setminus (H, S))^0$ is the union of finitely many maximal tails $M_1, \ldots, M_n$, with each M_i satisfying Condition (L) (i.e., every cycle with source in M_i has an exit) and strong CSP, with respect to a countable subset.

Theorem (Aljohani/Radler/Rangaswamy/Srivastava, 2021)

The following are equivalent for any graph E.

1. Every proper ideal of $L_K(E)$ is a product of strongly prime ideals.
2. E satisfies Condition (K) (i.e., each vertex which is the source of a cycle is the source of at least two distinct ones), and for every admissible pair (H, S), $(E \setminus (H, S))^0$ is either downward directed satisfying the strong CSP, or is the union of finitely many maximal tails satisfying strong CSP.

Intersections of Ideals

Theorem (Noether, 1921)

Every ideal in a commutative noetherian unital ring can be expressed as the intersection of finitely many irreducible, respectively primary, respectively *relatively-prime-indecomposable*, respectively *comaximally-indecomposable*, ideals.

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Theorem (Fuchs/Heinzer/Olberding, 2004)

Every ideal in a commutative unital ring can be expressed as the intersection of primal ideals.

Intersections of Prime-Power Ideals in LPAs

Theorem (2022)

Let E be a graph, let $m, r_1, \dots, r_m \in \mathbb{Z}^+$, and let $P_1, \dots, P_m$ be distinct prime ideals of $L_K(E)$. Then

$$P_1^{r_1} \cdots P_m^{r_m} = P_1^{r_1} \cap \cdots \cap P_m^{r_m}.$$

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Corollary

The following are equivalent for any graph E and proper ideal I of $L_K(E)$.

1. I is a product of prime (or prime-power) ideals.
2. I is an intersection of primary ideals.
3. I is an intersection of quasi-primary ideals.
4. I is an intersection of irreducible ideals.
5. I is an intersection of prime-power ideals.

Completely Irreducible and Strongly Prime Ideals in LPAs

Proposition (2022)

The following are equivalent for any graph E and proper ideal I of $L_K(E)$.

1. I is a product of completely irreducible ideals.
2. I is an intersection of finitely many completely irreducible ideals

Proposition (Aljohani/Radler/Rangaswamy/Srivastava, 2021)

The following are equivalent for any graph E and proper ideal I of $L_K(E)$.

1. I is a product of strongly prime ideals.
2. I is an intersection of finitely many strongly prime ideals.

Products vs. Intersections of Primes in LPAs

Example

Let E be the following graph.

If P is any nonzero prime ideal of $L_K(E)$, then P^2 is not the intersection of any collection of prime ideals of $L_K(E)$.

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Proof.

Recall that $L_K(E) \cong K[x, x^{-1}]$. Since $K[x, x^{-1}]$ is a PID, P is maximal, and $P^2 \neq P$.

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Proof.

Recall that $L_K(E) \cong K[x, x^{-1}]$. Since $K[x, x^{-1}]$ is a PID, P is maximal, and $P^2 \neq P$.

Suppose that $P^2 = \bigcap_{i \in Y} Q_i$, for some prime ideals Q_i of $K[x, x^{-1}]$. Then $Q_i \supseteq P$, for each i, and hence $Q_i = P$, as P is maximal. But then $P^2 = \bigcap_{i \in Y} P = P$ contradicts $P^2 \neq P$. □

When Products and Intersections of Primes Coincide

Theorem (2020)

The following are equivalent for any graph E.

1. For any $n \in \mathbb{Z}^+$ and prime ideals $P_1, \ldots, P_n$ of $L_K(E)$, we have
$$P_1 \cdots P_n = P_1 \cap \cdots \cap P_n.$$
2. Every proper ideal of $L_K(E)$ is semiprime.
3. Every ideal of $L_K(E)$ is of the form $I(H, S)$.
4. E satisfies Condition (K), i.e., every vertex in E which is the source of a cycle is the source of at least two distinct cycles.

Thank you!

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