On the Stability of Certain Classes of Semigroup Rings

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➦ R is an integral domain with quotient field K, I a nonzero ideal of R

- $I^{-1} = (R : I) = \{x \in K | xI \subseteq R\}$ called the inverse of I.
- $I^{-1} \cong Hom_R(I, R)$, so it is also called the dual of I.
- I is invertible if $IJ = R$ for some nonzero ideal of R, i.e., $II^{-1} = R$.
- $(I : I) = \{x \in K | xI \subseteq I$. $(I : I) \cong End_R(I) = Hom_R(I, I)$. It is the largest overring of R where I is an ideal.
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➡ Huneke and Weigand proved that for an abstract hypersurface R of dimension one (that is, a ring of the form $S/(f)$ where f is a prime element of the two-dimensional regular local ring S), and M and N R-modules, at least one of which has constant rank, if $M \otimes N$ is torsion-free, then either M or N is free. However, the hypothesis "hypersurface" cannot be changed to "complete intersection (that is a ring of the form $S/(f_1, \ldots, f_r)$, where S is a regular local ring and $(f_1, \ldots, f_r)$ is a regular sequence in the maximal ideal)". They provided the following example.

Example

Let $R = k[[X^4, X^5, X^6]] \cong \frac{k[[Y,Z,W]]}{(YW-Z^2, Y^3-W^2)})$, $I = (X^4, X^5)$ and $J = (X^4, X^6)$. Then R is a complete intersection, $I \otimes J$ is torsion-free, yet neither I nor J is free.

- For ideals I and J of a domain R, they proved that a necessary condition for $I \otimes J$ to be torsion-free is that $\mu_R(IJ) = \mu_R(I)\mu_R(J)$. However, the condition on the number of generators is not sufficient for $I \otimes J$ to be torsion-free as it is shown by the following example.

Example

Let $R = k[[X^4, X^5]]$, $I = (X^4, X^5)$ and $J = (X^8, X^{10})$. Then $\mu_R(IJ) = \mu_R(I)\mu_R(J)$, but $I \otimes J$ has torsion.

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- Finally, given a ring extension $R \subsetneqq S$ and an S-module N.Then $S \otimes_R N$ has two S-module structures. The "left" structure coming from the first factor and the "right" structure coming from the S-module structure on N. They proved that these structures are not even isomorphic. They illustrated this phenomenon with the following example in which the ring extension is actually birational.

Example

Let $R = k[[X^4, X^5]] \subsetneqq S = k[[X^4, X^5, X^6]]$, and let $J = X^4S + X^6S$. Then $J \otimes_R S \ncong S \otimes_R J$ as "left" S-modules. Moreover, if $I = X^4S + X^5S$, then $I \otimes_S J \otimes_R S$ and $I \otimes_S S \otimes_R J$ are not isomorphic as R-modules.

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- Conjecture 1.1 (Huneke-Wiegand conjecture). Let R be a Gorenstein local domain. Let M be a maximal Cohen-Macaulay R-module. If $M \otimes_R Hom(M, R)$ is torsionfree, then M is free.
- An ideal-theoretic version of the conjecture sustains that: (HW) If R is a one-dimensional Gorenstein local domain and I is a non-principal ideal of R, then $I \otimes_R Hom_R(I, R)$ has nonzero torsion.

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Example

Let k be a field and X an indeterminate over k. Let

$$S = k[[X^2, X^3]]$$
$$R = k[[X^3, X^5, X^7]]$$
$$m = (X^3, X^5, X^7)$$

Clearly, R is a one-dimensional local Noetherian domain with maximal ideal m. Further, one can check that $m^{-1} = (m : m) = S$ and $\overline{R} = k[[X]]$, $m \otimes_R m^{-1}$ has nonzero torsion.

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Let k be a field, X an indeterminate over k, $R = k[[X^3, X^4]]$, and $I = (X^4, X^6)$. Then R is a one-dimensional local Noetherian divisorial domain (hence Gorenstein) with maximal ideal $m := (X^3, X^4)$, and $D := (I : I) = k[[X^2, X^3]]$ is local with maximal ideal $M := (X^2, X^3) = X^2 k[[X]]$. Moreover, $I \otimes_R I^{-1}$ has nonzero torsion.

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- Let R be an integral domain and let M be an R-module. Then the trace of M is the ideal of R generated by the set $\{f(m)|f \in Hom_R(M, R), m \in M\}$.
- An ideal I of R is a trace ideal if I is the trace of some R-module, and R has the trace property (or is a TP-domain) if every ideal is a trace ideal. in [22], it was proved that the trace of I (as an R-module) is simply the product of I with its inverse I^{-1}, and R has the trace property if for every (nonzero) ideal I, either $II^{-1} = R$ or II^{-1} is a prime ideal of R.
- T. Lucas introduced several types of trace properties including TPP (trace property for primary ideals) and $PRIP$ (primary ideal Q such that Q^{-1} is a ring implies Q is prime). He provided an example of a Noetherian TP domain which is not a $PRIP$ domain.

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- An ideal $J \subseteq I$ in a commutative ring R is a reduction of I if $JI^n = I^{n+1}$ for some positive integer n.
- The notion of reduction was introduced by Northcott and Rees [41] to contribute to the analytic theory of ideals in Noetherian local rings with infinite residue field. The core of I, denoted $core(I)$, is the intersection of all reductions of I.
- The core was initially introduced by Sally [47] and appeared, among others, in the context of Briancon-Skoda's Theorem, which asserts that if R is regular with dimension d, then $core(I)$ contains the integral closure of I^d [31, Chapter 13].

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([46, Example 4.9]). Let k be an infinite field of positive characteristic p and $q \geq p$ be an integer not divisible by p. Consider the numerical semigroup ring $R = k[[X^{p^2}, X^{pq}, X^{pq+q}]]$, and let $I = \mathfrak{m}$ be the maximal ideal of R. Then R is a one-dimensional local Gorenstein domain, and $core(I) \supsetneqq (J^{n+1} : I^n)$ for any minimal reduction J of I.

- In Noetherian settings, the class of domains satisfying $core(I) = I^2 I^{-1}$ for all nonzero ideals lies strictly between the two classes of TP-domains and one-dimensional domains; and the equivalence holds in a large class of Noetherian domains. The next example features a one-dimensional Noetherian local domain with maximal ideal M such that $core(M) = M^3 \subsetneqq M^2 M^{-1}$.

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([33, Example 3.2]) Let k be a field and X an indeterminate over k. Let $R := k[[X^3, X^4]]$. Then, R is a one-dimensional Noetherian local domain with maximal ideal $M := (X^3, X^4)$ and hence $T := (M : M) = M^{-1} = k[[X^3, X^4, X^5]]$. Then $core(M) = M^3 \subsetneq M^2 = M^2T = M^2M^{-1}$.

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- An ideal I of R is called $\mathfrak{m}$-full if $(\mathfrak{m}I : x) = I$ for some $x \in \mathfrak{m}$, and I is said to be weakly $\mathfrak{m}$-full ideal provided that $(\mathfrak{m}I : m) \subseteq I)$, or equivalently, $(\mathfrak{m}I : m) = I$.
- Following is an example of weakly $\mathfrak{m}$-full ideals that are not $\mathfrak{m}$-full.

Example

([15, Example 2.7]) Let $R = \mathbb{C}[[X^4, X^5, X^6]]$. Then R is a one-dimensional complete intersection ring and $Q = (X^4)$ is a parameter ideal of R. Set $I = (Q : \mathfrak{m})$. Then I is weakly $\mathfrak{m}$-full but not $\mathfrak{m}$-full.

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- In [30], it is proved that if (R, M) is a local Noetherian domain such that M^{-1} is local with principal maximal ideal $N \neq M$ and $dim_{R/M}(M^{-1}/M) = 3$, then R has exactly three star operations.

Example

(2) ([30, Example 3.10]. Let k be a field, $R = k[[X^3, X^5, X^7]]$, and let M denote the maximal ideal of R. Then $M^{-1} = k[[X^2, X^3]] = R + RX^2 + RX^4$, $dim_{R/M} M^{-1}/M = 3$, and $N^3 \subseteq M$ but $N^2 \not\subseteq M$, where N is the maximal ideal of M^{-1}. So R has exactly 4 star operations

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- $SStable(R)$ is a multiplicative monoid.
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Thank you