Largeness of q-coefficients of q-binomial coefficients

Shashikant Mulay

Conference on Rings and Polynomials 2025
Graz, Austria

15'th of July 2025

Integer partitions

For a positive integer n, let $g(x) := (1-x)(1-x^2)\cdots(1-x^n)$.
For $0 \le r \le n$, the coefficient of x^r in $g(x)^{-1}$ and in $g(x)$ resp. are:

$$p(r) \quad \text{and} \quad Q_{even}(r) - Q_{odd}(r)$$

$p(r)$ is the count of all partitions of n and

$Q_*(r)$ counts the partitions of r in $*$ number of distinct parts .

Euler's famous "Pentagonal number theorem" :

$$Q_{even}(r) - Q_{odd}(r) = \begin{cases} (-1)^m & \text{if } r = \frac{m(3m-1)}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Integer partitions: post Euler

First the simply elegant Hardy-Ramanujan (1918) asymptotic:

$$p(n) \sim \frac{\exp(\pi\sqrt{2n/3})}{4n\sqrt{3}}$$

Now the fantastic Bruinier-Ono (2011) finite algebraic exact formula

$$p(n) = \frac{1}{24n-1} \sum_{Q \in \mathcal{Q}_n} P(\alpha_Q).$$

And yet'good' bounds for $p(n)$ are still sought-after !! Here are the currently sharpest bounds by Banerjee, Paule, Radu, Zeng (2023):

$$\frac{\exp(\pi\sqrt{2n/3})}{4n\sqrt{3}} \left(1 - \frac{1}{2\sqrt{n}}\right) < p(n) < \frac{\exp(\pi\sqrt{2n/3})}{4n\sqrt{3}} \left(1 - \frac{1}{3\sqrt{n}}\right).$$

Restricted partitions

For $N \in \mathbb{Z}_+$ and $a := (a_1, \ldots, a_m) \in \mathbb{Z}_+^m$, let ,

$$\Delta_a(n) := \{(i_1, \ldots, i_m) \in \mathbb{N}^m \mid i_1 a_1 + \cdots + i_m a_m = n\}$$

$$\Delta_a(N, n) := \{(i_1, \ldots, i_m) \in \Delta_a(n) \mid i_1 + \cdots + i_m \leq N\}$$

$$D_a(N, n) := |\Delta_a(N, n)| \quad \text{(Sylvester's \textit{denumerant})}$$

$$\sum D_a(N, n)\, x^n y^N = \frac{1}{(1 - yx^{a_1}) \cdots (1 - yx^{a_m})(1 - y)}.$$

$a_i \neq a_j \Rightarrow D_a(N, n) =$ no, of partitions of n in $\leq N$ parts $a_1, \ldots, a_m$.

Special case: $p(w; N, d) := D_{(1,2,\ldots,d)}(N, w)$ *i.e.*, the number of partitions of w in at most N parts and with each part at most d.
If $1 \leq w \leq \min\{N,\ d\}$, then $p(w; N, d) = p(w)$.

Gaussian binomial coefficients

Gauss gave us the q-binomial coefficients: for integers $0 \le k \le n$

$$\binom{n}{k}_q := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q^1)}$$

polynomials in q of degree $(n-k)k$ (the no. of k-dimensional subspaces of $\mathbb{F}_q^n$ for a prime-power q). Most remarkably,

$$\sum_{w=0}^{Nd} p(w;N,d)\, q^w = \binom{N+d}{d}_q.$$

Note that $p(w;N,d) = p(w;d,N)$ since (like the usual binomial coefficients),

$$\binom{n}{k}_q = \binom{n}{n-k}_q.$$

q-binomial coefficients

- Changing q to q^{-1} and multiplying by q^{Nd},

$$\sum_{w=0}^{Nd} p(w;N,d)\, q^{Nd-w} \;=\; q^{Nd}\binom{N+d}{d}_{q^{-1}} \;=\; \binom{N+d}{d}_{q}$$

  and hence $p(w;N,d) = p(Nd-w,N,d)$ (symmetry about $Nd/2$).

- Clearly, it suffices to restrict our attention to $w \le Nd/2$.
- Observe that for $\min\{N,\, d\} < w \le Nd/2$, the values of $p(w;N,d)$ are not related to $p(w)$ in any obvious way.

Aymptotics for $p(w; N, d)$

(1991) G. Almkvist and G. Andrews, J. Numb. Theory, *A Hardy-Ramanujan Formula for Restricted Partitions* :
Suppose $d,\, w \geq N$. Then, (as $N \longrightarrow \infty$)

$$p(w;\, N,\, d) \;\sim\; \theta\, (3 - 6\alpha + \alpha^2)\, \exp\left(\frac{-\alpha}{2}\right)$$

where $\alpha := \alpha(w; N, d)$ and $\theta := \theta(N, d)$ are defined as

$$\alpha \;:=\; \frac{3(Nd - 2w)^2}{Nd(N + d + 1)} \quad \text{and}$$

$$\theta \;:=\; \binom{N+d}{d} \sqrt{\frac{6}{\pi N d(N + d + 1)}} \left\{1 - \frac{1}{20}\left(\frac{1}{N} + \frac{1}{d} - \frac{1}{N + d + 1}\right)\right\}.$$

Semi-invariants

- Assume k is a field, $char\, k$ either 0 or $> N$. An N-ary *semi-invariant* (of *weight* w) is a polynomial $f \in k[z] := k[z_1, \ldots\ldots, z_N]$ such that

  (i) f is symmetric in $z_1, \ldots, z_N$,
  (ii) $f(z_1 + \alpha, \ldots, z_N + \alpha) = f(z_1, \ldots, z_N)$ for all $\alpha \in \overline{k}$.
  (iii) (f is homogeneous of degree w).

  N-ary semi-invariants form a subring of $k[z]$ isomorphic to $k^{[N-1]}$. Semi-invariants of weight $Nd/2$ are called *invariants*.

- **Definition**: For $N,\, d \in \mathbb{Z}_+$ define $H(w;\, N,\, d)$ to be the set of N-ary semi-invariants of weight w and z_i-degree $\leq d$ for $1 \leq i \leq N$.
  If Nd is even , then let $Inv(N, d) := H(Nd/2; N, d)$.
- As k-linear spaces $H(w;\, N,\, d) \,<\, k[z_1, \ldots\ldots, z_N]$.

Semi-invariants

- **Examples**: $\prod_{1\le i<j\le N}(z_i - z_j)^{2n} \in H(nN(N-1);\, N,\, 2n(N-1))$. In fact, this is an invariant. $\sum_{1\le i<j\le N}(z_i - z_j)^{2n} \in H(2n;\, N,\, 2n)$ is not an invariant when $N \neq 2n^2$.
- $H(w,\, 1,\, d)$, $H(1,\, N,\, d)$, $H(2n+1,\, 2,\, d)$ $H(> Nd/2,\, N,\, d)$ are 0. $H(2n, 2, d)$ is 1 dimensional.
- $Inv(N) := \cup_d Inv(N, d)$ is an $N-2$ dimensional subring of $k[z]$. P. Gordon and D. Hilbert: $Inv(N)$ is a finitely generated ring over k.
- Pairs (N, d) with $Inv(N, d) \neq 0$ were (finally) determined in 1985: J. Dixmier: *Quelques résultats et conjectures concernant les séries de Poincaré des invariants des formes binaires.*

Cayley and Sylvester

- For this talk, sequence $c_0, c_1, \ldots, c_n$ is *unimodal* if $c_i \leq c_{i+1}$ for $0 \leq i < n$ and *strictly unimodal* if $c_i < c_{i+1}$ for $0 \leq i < n$.
- In 1852-53, Cayley claimed (without proof) that for $k = \mathbb{C}$, $N \geq 2$ and $w \leq Nd/2$, the vector space $H(w;\, N,\, d)$ has dimension
$$p(w; N, d) - p(w-1, N, d).$$
So, in particular, $p(0, N, d), \ldots\ldots, p(\lfloor Nd/2 \rfloor, N, d)$ is unimodal.
- Ultimately, Cayley's claim was proved by Sylvester in his famous 1878 paper: *Proof of the hitherto undemonstrated fundamental theorem of invariants.* Subsequently, Sylvester built (in several papers) what G. Andrews calls "the modern theory of partitions".
- Yet, $p(w; N, d)$ as well as $p(w; N, d) - p(w-1, N, d)$ remain un-understood !

Post Sylvester

- More than 100 years after Sylvester came the influential UC Berkeley Ph. D. thesis of K. M. O'Hara : (KOH) *Unimodality of Gaussian coefficients: a constructive proof*, J. Combin. Theory Ser. A 53:1 (1990) D. Zeilberger calls it a "magnificent combinatorial proof of the unimodality" Most of the recent work on $p(w; N, d)$ takes inspiration from KOH.

- Unimodality re-proved with S_n-representation-theoretic view in (2010) I. Pak - E. Vallejo, SIAM J. Discrete Math. 24
  Reductions of Young tableau bijections .

- **Strict unimodality**: (2013) I. Pak - G. Panova: Comp. Rend.
  $p(w; N, d) - p(w-1, N, d) \geq 1$ provided $N,\ d \geq 8$.

Beyond unimodality

- Improved unimodality (2014) Vivek Dhand: Discrete Math.
  A combinatorial proof of strict unimodality for q-binomial coefficients
  If $8m \leq \min N, d$, then $p(w; N, d) - p(w-1, N, d) \geq m$ for $2m \leq w \leq Nd/2$. Simialr inequality with a lower bound on w that is quadratic in m by F. Zanello in (2015).

- Exponential bound (2017) by I. Pak - G. Panova: J. Comb. Th. (A)
  Bounds on certain classes of Kronecker and q-binomial coefficients
  PP lower bound: Assume $\min\{N,\, d\} \geq 8$ and $0 \leq w \leq \lfloor Nd/2 \rfloor$. Let $M := \min\{2w,\, N^2,\, d^2\}$. Then,

$$p(w; N, d) - p(w-1, N, d) \;\geq\; \frac{2^{\sqrt{M}}}{250 M^{9/4}}.$$

Weight sensitive lower bounds

- Since $p(w; N, d) = p(w; d, N)$, without loss assume $d \geq N$.
- Recall PP lower bound: If $N \geq 8$ and $0 \leq w \leq \lfloor Nd/2 \rfloor$, then

$$p(w; N, d) - p(w-1, N, d) \; \geq \; \left\lceil \frac{2^{\sqrt{M}}}{250 M^{9/4}} \right\rceil ; \quad M := \min\{2w, \, N^2\}.$$

- For $2w \geq N^2$, the PP lower bound is independent of $(w, \, d)$!
  Also, the case $3 \leq N \leq 7$ needs attention ! For example,

$$p(100; 3, 100) - p(99, 3, 100) \; = \; 17.$$

So, the quest for better bounds is still in its early stage !

New weight sensitive lower bounds

- Enter our lower bounds: SM (2021)

  Enumerative Combinatorics and Applications,

  Semi-Invariants of Binary Forms and Symmetrized Graph-Monomials

- **Virtue**: for fixed N, as $w \longrightarrow \infty$ our lower bounds $\longrightarrow \infty$ while PP remains constant since $\min\{2w, N^2\}$ stays fixed.

- **Drawback**: our bounds work only for a range (depending on d) of values of w and for w in that range they are independent of d. In particular, we fail to fully recover the previous results !

Our lower bounds

Assume $d \geq N \geq 3$ and $w \leq Nd/2$. Let $a : a_1 < \cdots < a_{s+1}$ be a seq. of pos. int. with $s \geq 1$ and $a_1 + \cdots + a_{s+1} = N$ (so $s \leq -1 + (\sqrt{8N+1} - 1)/2$). Define

$$\begin{aligned} wt(N,a) &:= \frac{N^2 - a_1^2 - \cdots - a_{s+1}^2}{2} \\ \delta(N,d,a) &:= a_1(d + a_1 - N - 1) + wt(N,a) + \left\lfloor \frac{1}{a_1} \right\rfloor \end{aligned}$$

Then, for $wt(N,a) + 1 \leq w \leq \delta(N,d,a)$,

$$p(w;N,d) - p(w-1,N,d) \geq \binom{w - wt(N,a) + s - 1}{s-1}.$$

Max value of $wt(N, a)$

For $s \in \mathbb{Z}_+$ and partition $a : a_1 < \cdots < a_{s+1}$ of N,
we have $wt(N, a) \leq \varpi(s, N)$, where

$$
\begin{aligned}
\varpi(s, N) \;&=\; \frac{(s+1)(s+2)}{2}\left\lfloor \frac{N}{s+1} - \frac{s}{2} \right\rfloor^2 \\
&+ \frac{(s+1)^2(s+2) - 2N(s+2)}{2}\left\lfloor \frac{N}{s+1} - \frac{s}{2} \right\rfloor \\
&+ \frac{3(s+1)^4 + 2(s+1)^3 - 3(1+4N)(s+1)^2}{24} \\
&+ \frac{24N^2 - 2(1+6N)(s+1)}{24}.
\end{aligned}
$$

Our lower bounds: an example

Let $N = 15 \leq d$, $wt(a) := wt(15, a)$ and $\delta(a) := \delta(15, d, a)$.

Here PP lower bound is just 1.

$a : 3 < 4 < 8$: $wt(a) = 68$, $\delta(a) = 3d + 29$ and $69 \leq w \leq 3d + 29$

$$p(w; 15, d) - p(w-1; 15, d) \ \geq \ w - 67.$$

$a : 2 < 3 < 4 < 6$: $wt(a) = 80$, $\delta(a) = 2d + 52$ and $81 \leq w \leq 2d + 52$

$$p(w; 15, d) - p(w-1; 15, d) \ \geq \ \frac{w^2}{2} - \frac{157w}{2} + 3081.$$

$a : 1 < 2 < 3 < 4 < 5$: $wt(a) = 85$, $\delta(a) = d + 71$ and $86 \leq w \leq d + 71$

$$p(w; 15, d) - p(w-1; 15, d) \ \geq \ \frac{w^3}{6} - \frac{83w^2}{2} + \frac{10333w}{2} - 95284.$$

Lower bounds versus actual values

Comparison of values of $p(w; 45, 55) - p(w-1; 45, 55)$ and lower bounds:

w	872	873	875
PP	759	770	792
SM	376992	435897	575757
value	121307660637674779775810	123136493996785875153133	126854496384791530573137

Let $f(d) := p(900; 45, d) - p(899; 45, d)$ $(d \geq 75)$.
PP $= 1121$ a 4-digit number and SM $= 10295472$ an 8-digit number.

$$\begin{aligned} f(76) &= 10695952533979786987999095 \quad 26 \text{ digits} \\ f(78) &= 13150261598814599756745952 \quad 26 \text{ digits} \\ f(80) &= 15869693093392541521308815 \quad 26 \text{ digits} \end{aligned}.$$

Lower bounds computation

- Each SM lower bound in the second row of the table just shown corresponds to the same $a : 3 < 4 < 5 < 6 < 7 < 8 < 12$.
- The SM lower bound for $w = 900$, $N = 45$ and $d \geq 75$ shown above, corresponds to

$$a : 1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$$

- For $(w,\ N) = (900,\ 45)$, as d ranges from 45 to 74, the largest SM lower bound varies from 1 to 10295472 while the PP lower bound remains fixed.

Method of investigation in SM (2021)

Given a partition $a : a_1 < \cdots < a_{s+1}$ of N in $s+1$ distinct parts, and $wt(N,a)+1 \leq w \leq \delta(N,d,a)$ we **construct** as many linearly independent semi-invariants of weight w as our lower bound.

Assume $N \geq 2$ and either $char\, k = 0$ or $char\, k > N$
Recall *symmetrization* operator $sym : \; k[z] \longrightarrow k[z]$:

$$sym(P(z_1,\ldots,z_N)) \;:=\; \sum_{\sigma\in S_N} P(z_{\sigma(1)},\ldots\ldots,z_{\sigma(N)}).$$

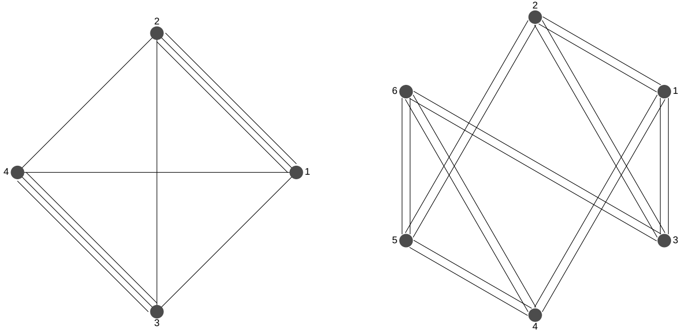
Then, sym maps $k[z_2-z_1,\, z_3-z_1,\, \ldots\, z_N-z_1]$ to itself.

Theorem: $f \in k[z]$ is a semi-invariant if and only if

$$f \;=\; sym(g) \quad \text{for some} \;\; g \in \; k[z_2-z_1,\, z_3-z_1,\, \ldots\, z_N-z_1].$$

Multigraphs

Multigraph: graph in which there may be 2 or more edges connecting a pair of vertices. (N, d)-*multigraph*: undirected loop-less multigraph on N vertices each of degree $\leq d$.



Graph-monomials

Let $symm(N, d)$ be the set of all $N \times N$ zero diagonal symmetric matrices with entries in $\mathbb{N}$ and such that each row / column sum is at most d. A **labeled** (N, d)-multigraph Γ, is determined uniquely by its *adjacency matrix* $A(\Gamma) \in symm(N, d)$.

For $A \in symm(N, d)$, define $weight(A)$ to be half the sum of its entries. If $A := [a_{ij}]$, define $mon(A) \in k[z_2 - z_1, \ldots\ldots, z_N - z_1]$ by

$$mon(A) \; := \; \prod_{1 \leq i < j \leq N} (z_i - z_j)^{a_{ij}}.$$

Graph-monomial of Γ (introduced by J. Petersen in 1890 s) is $mon(A(\Gamma))$.

$sym(mon(A))$ depends only on the permutation-conjugacy class of A
$A \in symm(N, d)$ has $weight(A) = w$ iff $sym(mon(A))$ is in $H(w;\, N,\, d)$;
$weight(A) = Nd/2$ iff $sym(mon(A))$ is in $Inv(N,\, d)$.

Symmetrized graph-monomials

- $mon(A) \neq 0$ but $sym(mon(A))$ is very likely to be 0 !
- *e.g.*, out of 2274 isomorphisms classes of $(6, 10)$-regular multigraphs, only 1137 have a nonzero symmetrized graph-monomial.
- To the best of our knowledge, the only investigation of this issue is (1992) G. Sabidussi:

  Binary invariants and orientations of graphs, Discrete Math. 101.

  It relates nonzero-ness of the symmetrization to orientation preserving automorphisms of the multigraph. This relationship is not of much practical use (as the author himself indicates !)

A criterion for nonzero symmetrization

Suppose $A = A(\Gamma)$ where Γ is an (N, d)-multigraph where $N \geq 3$.

A main results of SM (2021) ECA gives a usable sufficient condition on A for $sym(mon(A)) \neq 0$.The gist of this condition is as follows:

If the vertices of Γ can be partitioned into parts of sizes $a_1 < \cdots < a_{s+1}$ such that

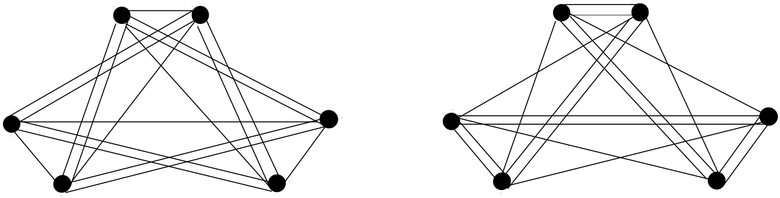
(i) each $a_i \times a_i$ part has a nonzero symmetrized graph-monomial and

(ii) the number of edges within each part is small compared to the total number of edges,

then the symmetrization of the graph-monomial of Γ is nonzero.

Independence of symmetrized graph-monomials

- The symmetrized graph-monomials of two non-isomorphic multigraphs may be nonzero multiples of each other.
  For example, this is indeed the case for multigraphs:



- In SM (2021), we do produce linearly independent symmetrized graph-monomials.

A sample of open questions

- Find a stronger sufficient condition on A that ensures $sym(mon(A)) \neq 0$. SM (2021) gives a test for 'stronger' : namely, recovery of Hermite's skew invariant !
- What conditions on a collection of multigraphs ensure that their symmetrized graph-monomials are linearly independent ?
- Some more questions of this type in an appendix of MQS (2018);

  Strong Fermion Interactions in Fractional Quantum Hall States.
- Find (w, d)-sensitive lower bounds for $p(w; N, d) - p(w-1; N, d)$; at least when Nd is even and $w = Nd/2$. Fix (N, d) and determine intervals of w where $p(w; N, d) - p(w-1; N, d)$ is monotonic.

Back to where we started !

Recall our favourite polynomial $g(x) = (1-x)(1-x^2)\cdots(1-x^n)$.
Let $\psi_n(r) :=$ coeff. of x^r in $g(x)$, *i.e.*, $Q_e^*(r) - Q_o^*(r)$
where $*$ indicates that each part is at most n.

Euler's penagonal number theorem evaluates $\psi_n(r)$ for $0 \le r \le n$.
What is the value of $\psi_n(r)$ for $n+1 \le r \le n(n+1)/2$? Easily,

$$|\psi_n(r)| \le \binom{n-1+r}{n-1} \quad \text{for all } r \in \mathbb{N}.$$

Since $g(x) = (-1)^n \cdot x^{n(n+1)/2} \cdot g(1/x)$,

$$\psi_n(r) = (-1)^n \cdot \psi_n\left(\frac{n(n+1)}{2} - r\right).$$

So, suffices to know $\psi_n(r)$ for $n+1 \le r \le n(n+1)/4$.

Hard formula

Here is a formula for $\psi_n(r)$: given a positive integer k, define

$$\sigma_n(k) := \sum_{1 \le d \le n,\ d \mid k} d, \quad \text{and} \quad \alpha_n(k) := \frac{\sigma_n(k)}{k}.$$

Then, $\psi_n(r)$ is the sum:

$$\sum_{i_1 + 2i_2 + \cdots + m i_m = r} \frac{(-1)^{i_1 + i_2 + \cdots + i_m}}{i_1! i_2! \cdots i_m!} \alpha_n(1)^{i_1} \alpha_n(2)^{i_2} \cdots \alpha_n(m)^{i_m}.$$

Mysteries of $\psi_n(r)$

- Indeed $\psi_n(r) = 0$ if $n = 3\, mod\, 4$ and $r = n(n+1)/4$.
  What are the $(n,\, r)$ for which $\psi_n(r) = 0$?
- What are the pairs $(n,\, r)$ for which $\psi_n(r) > 0$ (resp. < 0) ?
- Fixing n, determine r with $\psi_n(r)$ (resp. $|\psi_n(r)|$) maximal.
- As $n \longrightarrow \infty$, does $\max\{\psi_n(r)\} - |\min\{\psi_n(r)\}| \longrightarrow \infty$?