Primes and absolutely or non-absolutely irreducible elements in atomic domains

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Conference on Rings and Polynomials, TU Graz

July 17, 2025

Outline

- Absolute irreducibility
- Motivation
- New Results

Absolute irreducibility

Definition 1

Let R be a commutative ring with identity.

1. A non-zero non-unit $r \in R$ is said to be **irreducible** in R if whenever $r = ab$, then either a or b is a unit.

2. An irreducible element $r \in R$ is called **absolutely irreducible** if for all natural numbers n, every factorization of r^n is essentially the same as $r^n = r \cdots r$, e.g., in

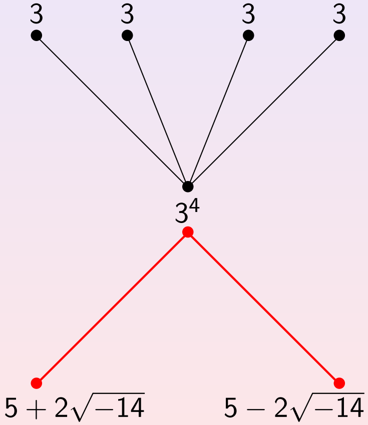
$$\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[x] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\},$$

$$\binom{x}{n} = \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}$$ (Rissner, Windisch, 2021).

3. If r is irreducible but there exists a natural number $n > 1$ such that r^n has other factorizations essentially different from $r^n = r \cdots r$, then r is called **non-absolutely irreducible**.

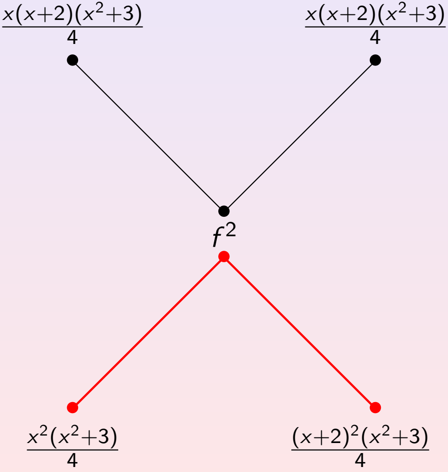
Examples of non-absolutely irreducible elements

In $\mathbb{Z}[\sqrt{-14}]$



Examples of non-absolutely irreducible elements

Consider $f = \frac{x(x+2)(x^2+3)}{4} \in \mathrm{Int}(\mathbb{Z})$.



- **See** (N, 2020) **for general constructions of non-absolutely irreducibles in** $\mathrm{Int}(\mathbb{Z})$.
- **Open Problem:**
  A complete characterization of the absolutely irreducible elements of $\mathrm{Int}(\mathbb{Z})$**.**

Motivation I

Recall: Let D be a domain with quotient field K. Then

$$\mathrm{Int}(D) = \{f \in K[x] \mid f(D) \subseteq D\}.$$

Theorem 1 (Frisch, 2013, Frisch, N., Rissner, 2019)

Let D be a Dedekind domain such that;

1. D has infinitely many maximal ideals and
2. $|D/M| < \infty$ for each maximal ideal M.

Let $1 < m_1 \leq m_2 \leq \cdots \leq m_n \in \mathbb{N}$. Then there exists a polynomial $H \in \mathrm{Int}(D)$ with exactly n essentially different factorizations of lengths $m_1, \ldots, m_n$.

Motivation I

(Frisch, 2013, Frisch, N., Rissner, 2019) Given any finite multi-set of integers greater than one, say $\{2, 3, 5, 5\}$, there exists $H \in \text{Int}(D)$ such that

$$\begin{aligned} H &= h_1 \cdot h_2 \\ &= f_1 \cdot f_2 \cdot f_3 \\ &= g_1 \cdot g_2 \cdot g_3 \cdot g_4 \cdot g_5 \\ &= \ell_1 \cdot \ell_2 \cdot \ell_3 \cdot \ell_4 \cdot \ell_5. \end{aligned}$$

Such constructions require absolutely irreducible elements.

Motivation II

Non-absolutely irreducible elements are necessary for studying patterns of factorizations. For instance factorization schemes.

Definition 2 (Definition 9.2.1, Geroldinger & Halter-Koch, 2006)

Let H be an atomic monoid, $m, r \in \mathbb{N}$ and

$$N = (n_{j,i})_{(j,i)\in[1,m]\times[1,r]} \in \mathbb{N}_0^{m\times r}$$

an (m, r)-matrix of non-negative integers. An element $c \in H$ is said to admit the factorization scheme N if there exist distinct irreducible elements $a_1, \ldots, a_r$ such that

$$c = a_1^{n_{i,1}} \cdots a_r^{n_{i,r}} \text{ for all } i \in [1, m]$$

Motivation III

Theorem 2 (Chapman and Krause, 2012)

Let $\mathcal{O}_K$ be the ring of integers of a number field.

1. An element $r \in \mathcal{O}_K$ is absolutely irreducible if and only if (r) is a minimal power of a prime ideal.
2. For each nonzero nonunit $r \in \mathcal{O}_K$, there exists a sequence $r_1 \cdots r_t$ of absolutely irreducible elements and a minimal $m \in \mathbb{N}$ such that
$$r^m = r_1 \cdots r_t$$
where this representation by atomic decay is unique up to ordering and associates for $r_1 \cdots r_t$.

Corollary 1 (Chapman and Krause, 2012)

$\mathcal{O}_K$ is a unique factorization domain if and only if every irreducible element of $\mathcal{O}_K$ is absolutely irreducible.

New Results

Theorem 3 (Fadinger, Frisch, N., Smertnig, Windisch, 2025)

There exists a Dedekind domain that is not a unique factorization domain and such that all of its irreducible elements are absolutely irreducible but none of them prime.

Theorem 4 (Fadinger, Frisch, N., Smertnig, Windisch, 2025)

There exists a Dedekind domain that is not a unique factorization domain, contains a prime element, and such that all of its irreducibles elements are absolutely irreducible.

New Results

1. (Fadinger, Frisch, N., Smertnig, Windisch, 2025) Atomic domains showing the following eight scenarios. **+** means existence and - means non-existence.

	Non-absolutely irreducible	**Absolutely irreducible but not prime**	**Prime**
$\mathrm{Int}(\mathbb{Z})[y]$	$+$	$+$	$+$
$\mathrm{Int}(\mathcal{O}_K)$	$+$	$+$	-
$\mathbb{R}+\mathbb{C}[X]$	$+$	-	$+$
$\mathbb{R}+\mathbb{C}[[X]]$	$+$	-	-
Certain Dedekind domains with class group $\mathbb{Z}^n$	-	$+$	$+$
Certain Dedekind domains with class group $\mathbb{Z}^n$	-	$+$	-
UFDs	-	-	$+$
Fields	-	-	-

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