Pure Braid Group Presentations via Longest Elements

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Algebra

- Consider a Dynkin graph of type $A_n := \underset{1}{\bullet} - \underset{2}{\bullet} - \cdots - \underset{n-1}{\bullet} - \underset{n}{\bullet}$ with n vertices. There is an associated braid group

$$\mathrm{Br}A_n := \left\langle s_1, \ldots, s_{n-1} \;\middle|\; \begin{array}{l} s_i s_j = s_j s_i \text{ if } |i-j| \geq 2, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for all } i = 1, 2, \ldots, n-2 \end{array} \right\rangle.$$

- The Weyl group $\mathrm{W}A_n$ associated to A_n has a coxeter presentation

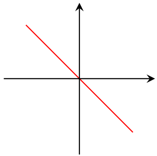
$$\mathrm{W}A_n := \left\langle s_1, \ldots, s_{n-1} \;\middle|\; \begin{array}{l} s_i s_j = s_j s_i \text{ if } |i-j| \geq 2, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for all } i = 1, 2, \ldots, n-2 \\ s_i^2 = 1 \end{array} \right\rangle.$$

Definition 1

The pure braid group $\mathrm{PBr}A_n$ *associated to* A_n *is the kernel of the surjective group homomorphism* $f\colon \mathrm{Br}_{A_n} \to \mathrm{W}A_n$

Topology

- Consider the $A_2 \subseteq \mathbb{R}^2$ root system given by



$x = 0$

$y = 0$

$x + y = 0$

- A hyperplane H is a subspace whose dimension is one less than that of its ambient space V.
- A hyperplane arrangement is a finite collection of hyperplanes.
- A chamber is a connected component of $V \setminus H$
- An arrangement is called Coxeter if it arises as the set of reflection hyperplanes of a finite real reflection group.

- Complexifying the real root systems, induces a group action of a group G on the hyperplane arrangement

Definition 2

The pure braid group $\mathrm{PBr} \cong \pi_1(\mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}})$ *which is the fundamental group of the complexified complement.*

Notation

- Let $\mathcal{A} \subseteq A_n$, $\mathcal{A}$ connected. Consider the longest element (in the symmetric group) over $\mathcal{A}$ denoted by $\ell_{\mathcal{A}}$

- Over A_n, $\ell_{A_n} = s_1(s_2 s_1) \cdots (s_n s_{n-1} \cdots s_3 s_2 s_1)$

Example 1

Consider $A_2 := \underset{1}{\bullet} - \underset{2}{\bullet}$, *then*

$$\textit{connected subgraphs} = \{\ \bullet\circ, \circ\bullet, \bullet\bullet\ \}$$

$$\textit{longest elements} = \{\ \ell_{\bullet\circ} = s_1, \ell_{\circ\bullet} = s_2, \ell_{\bullet\bullet} = s_1 s_2 s_1\ \}$$

Proposition 1 (Namanya, 2023)

Let $\mathcal{A}$ be a connected subgraph of the A_n Dynkin graph, then

1. *the set* $\{\ell_{\mathcal{A}}^2\}$ *generates* PBr_{A_n}.
2. *there is a new presentation for the pure braid group.*

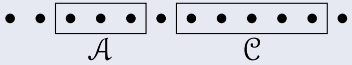
Relations ?

Definition 3

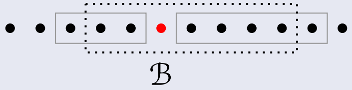
Let $\mathcal{A}, \mathcal{B}$ be connected subgraphs of the graph A_n. If $\mathcal{A} \cap \mathcal{B} \neq \emptyset$, then define the distance $d(\mathcal{A}, \mathcal{B}) = 0$. Else, $d(\mathcal{A}, \mathcal{B})$ is defined to be the number of edges between $\mathcal{A}$ and $\mathcal{B}$.

Relations continued

The case $d(\mathcal{A}, \mathcal{C}) = 2$ corresponds to when $\mathcal{A}$ and $\mathcal{C}$ are precisely one node apart, namely



Given such a pair, we say that a subgraph $\mathcal{B}$ is compatible with $(\mathcal{A}, \mathcal{C})$ if $\mathcal{B}$ is a connected subgraph of the following dotted area, containing the red node.



Theorem 1 (Namanya, 2023)

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be connected subgraphs of the graph A_n, as a slight abuse of notation, write $\mathcal{A} := \ell^2_{\mathcal{A}}$. The pure braid group PBr_{A_n} has a presentation with generators given by connected subgraphs $\mathcal{A} \subseteq A_n$, subject to the relations

1. $\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$ *if* $d(\mathcal{A}, \mathcal{B}) \geq 2$, *or* $\mathcal{A} \subseteq \mathcal{B}$, *or* $\mathcal{B} \subseteq \mathcal{A}$.
2. *For all* $\mathcal{A}$ *and all* $\mathcal{C}$ *such that* $d(\mathcal{A}, \mathcal{C}) = 2$, *then*

$$(\mathcal{A} \cup \mathcal{B})^{-1} \cdot (\mathcal{A} \cdot \mathcal{B} \cdot \mathcal{C}) \cdot (\mathcal{B} \cup \mathcal{C})^{-1} = (\mathcal{C} \cup \mathcal{B})^{-1} \cdot (\mathcal{C} \cdot \mathcal{B} \cdot \mathcal{A}) \cdot (\mathcal{B} \cup \mathcal{A})^{-1}$$

for all $\mathcal{B}$ *with compatible* $(\mathcal{A}, \mathcal{C})$.

Example 2

Consider $A_3 := \underset{1}{\bullet} - \underset{2}{\bullet} - \underset{3}{\bullet}$, *then*

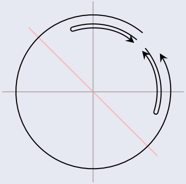
$$\text{connected subgraphs} = \{ \bullet\circ\circ, \circ\bullet\circ, \circ\circ\bullet, \bullet\bullet\circ, \circ\bullet\bullet, \bullet\bullet\bullet \}$$

$$\text{longest elements} = \{s_1, s_2, s_3, s_1s_2s_1, s_2s_3s_2, s_1s_2s_1s_3s_2s_1\}$$

$$\mathrm{PBr}_{A_3} := \left\langle a, b, c, d, e, f \;\middle|\; \begin{array}{l} ac = ca, af = fa, ad = da, bd = db \\ be = eb, bf = fb, ec = ce, fc = cf \\ d^{-1}abce^{-1} = e^{-1}cbad^{-1} \end{array} \right\rangle .$$

Remarks

1. The $\ell^2_{\mathcal{A}}$ are motivated by



2. This new presentation of the PBr_{A_n} answers a question of (Donovan and Wemyss, 2019).
3. The pure braid groups of other Coxeter arrangements are not in general generated by the analogue of $\ell^2_{\mathcal{A}}$.

W. Donovan and M. Wemyss, *Twists and braids for general 3-fold flops*, J. Eur. Math. Soc. (JEMS), **21** (2019), no. 6, 1641–1701.

C. Namanya, *Pure braid group presentations via longest elements*, J. Algebra, **628**, (2023) 1–21.

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Thank you for your attention!