Gröbner Bases Native to 'pseudo'-Hodge Algebras, with Application to the Algebra of Minors

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- Gröbner bases are well-suited to both of the above!
- Gröbner bases give theoretical insight as well as are the key tool in effective methods

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- Gröbner bases do not naturally *preserve symmetry*!

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$$W_n := \mathbb{C}\left[\left\{X_i, \frac{\partial}{\partial X_i}\right\}_{i\in[n]}\right] \Big/ \left\langle \left\{ \frac{\partial}{\partial X_i}\cdot X_i - X_i \cdot \frac{\partial}{\partial X_i} - 1 \right\}_{i\in[n]} \right\rangle$$

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Question

Develop a Gröbner basis theory which takes advantage if variety corresponding to ideal has large symmetry group, or is 'determinantal'

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- If A is an ASL, the product of two standard monomials can be straightened into a linear combinaton of 'smaller' standard monomials
- ASLs arise as coordinate rings of algebraic varieties, e.g. Grassmanians, determinantal varieties, flag varieties, Schubert varieties

Bideterminants (products of minors)

- Example of Hodge algebra - algebra of bideterminants

  e.g. $A = \mathbb{F}[X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}, Y] \Big/ \langle X_{1,2}X_{2,1} - X_{1,1}X_{2,2} + Y\rangle$

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- The above generalizes -
  - poly ring with one variable for each minor of $n \times m$ matrix
  - quotient by relations between minors
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- Advantage - smaller expressions for 'determinant-like' polynomials; bideterminants are reflect symmetries coming from the action (representation theory) of GL_n

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Question

Can we build a theory of Gröbner bases 'native' to p-ASLs, i.e. Gröbner bases without referencing ideal J?

Challenges

- Basis of p-ASL $A = {}^{\mathbb{F}[\vec{X}]} \big/ {}_{J}$ consists only of standard monomials (monomials outside Σ), not all monomials in $\vec{X}$

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- Product of standard monomials not necessarily standard, might require straightening
  - How do you define term order?
  - How would you define division of monomials?
  - What plays the role of monomial ideals?

Term Order & Division

- A p-ASL term order on a p-ASL A is a total order $\prec$ on standard monomials in A such that
  - $1 \preceq m$
  - If $a \prec b$ and $c \preceq d$, and $ac, bd \neq 0$, then
$$\mathrm{LM}(ac) \prec \mathrm{LM}(bd)$$

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$$\mathrm{LM}(ac) \prec \mathrm{LM}(bd)$$
- When does standard monomial m divide m':
  - ordinary division in the polynomial ring, or
  - m divides m' if there exists standard monomial f such that
$$\mathrm{LM}(mf) = m'$$

Auxilliary Algebra of Leading Terms

- Given p-ASL A, algebra of leading terms w.r.t. A is another p-ASL A_{lt} on the same variables, and the same standard monomials such that for standard monomials m, m'

$$\pi_{\mathrm{lt}}(m) \cdot \pi_{\mathrm{lt}}(m') = \overbrace{0}^{\text{no straightening}} \quad \text{or} \quad \underbrace{\pi_{\mathrm{lt}}(\mathrm{LT}(mm'))}_{\text{leading term of straightening}}$$

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Proposition

Every p-ASL A admits two algebras of leading terms – A_{gen} where the product is never 0, and, A_{disc} where product is 0 unless mm' is also a standard monomial.

Definition of p-ASL Gröbner Basis

- Given p-ASL A, algebra of leading terms A_{lt}, and an ideal $I \subseteq A$, then $G \subseteq A$ is a p-ASL Gröbner basis if:
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  - $\langle\{\pi_{lt}(LM(g)) : g \in G\}\rangle = \langle\{\pi_{lt}(LM(f)) : f \in I\}\rangle$ (standard monomial ideals in A_{lt})

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Universal	✓	✓
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Corollary (Grochow-N, 2025)

The algebra of bideterminants has a p-ASL term order, thus we have a Gröbner basis theory (called bd-Gröbner bases).

Applications to Bideterminant Algebra

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Takeaway

1. *Given all our machinery, the proof is one-line*
2. *In the ordinary case, universal Gröbner basis are known only for maximal minors and minors of size 2*

Future Work

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- Get $\mathfrak{bd}$-Gröbner bases of annihilating D-ideals
- Compute Weyl closure, b-functions, etc. using bd-Gröbner bases in the Weyl algebra
- See if we can develop a bipermanent Gröbner basis theory (codimension of singular locus of permanent hypersurface is unknown!)

References

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