Invariant subspaces

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- Based on the paper: *"Relative Cancellation"*, 2025.

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CHP

Neena Gupta, *The Zariski Cancellation Problem and Related Problems in Affine Algebraic Geometry*, 2022

Goal

To give a characterization of the polynomial ring using the concept of invariant subspaces.

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I would like to thank Mesyan for suggesting the word 'trivial' here instead of 'proper'.

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K is algebraically closed and of characteristic zero.

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$V_d :=< 1, x, \cdots, x^d >$, for some $d \geq 1$, the set of all polynomials of degree at most d.

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Example: $K[x_1, \cdots, x_n]$

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Conjecture: If A is commutative, Condition (ii) is superfluous.

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- It is regular 3-dimensional. We know these days it is not isomorphic to a polynomial ring.
- If one shows that $A[x]$ is isomorphic to the polynomial ring with four variables, she/he has solved ZCP.
- Can $A[x]$ have nontrivial invariant subspaces?

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- Which Leavitt path algebras have no nontrivial invariant subspaces?

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