THREE DEFINITIONS OF KRULL MODULES AND RELATIONSHIP AMONG THEM

Mu'amar Musa Nurwigantara
Universitas Gadjah Mada
muamar.musa.n@ugm.ac.id

Conference on Rings and Polynomials
Technische Universität Graz
14th-19th of July 2025

Outline

1. Preliminaries
2. Krull Modules in the sense of Wijayanti
3. Strongly Krull Modules
4. Krull Modules in the Sense of Costa and Johnson
5. Relationship Between Them
6. BIBLIOGRAPHY

Let D be an integrally closed domain and M be a torsion-free D-module.

Theorem 1

There exists a field K such that D is embedded into K, namely

$$K = \left\{ \frac{r}{s} \mid r \in D, s \in D \backslash \{0\} \right\},$$

the set of all equivalence classes in $D \times D \backslash \{0\}$ built by the relation $(r_1, s_1) \sim (r_2, s_2)$ iff $s_2 r_1 = s_1 r_2$, i.e. $\frac{r_1}{s_1} = \frac{r_2}{s_2}$ iff $s_2 r_1 = s_1 r_2$.
(K is called **quotient field**/**field of quotients**/**field of fractions**.)

Theorem 2

There exists a K-vector space V such that M is embedded into V, namely

$$V = \left\{ \frac{m}{s} \mid m \in M, s \in D \backslash \{0\} \right\},$$

the set of all equivalence classes in $M \times D \backslash \{0\}$ built by the relation $(m_1, s_1) \sim (m_2, s_2)$ iff $s_2 m_1 = s_1 m_2$, i.e. $\frac{m_1}{s_1} = \frac{m_2}{s_2}$ iff $s_2 m_1 = s_1 m_2$.
Since $K \cdot M = V$, denote V as KM.
(KM is called **quotient module**/**module of quotients**/**module of fractions**).

Let D be an integrally closed domain and M be a torsion-free D-module.

Theorem 1

There exists a field K such that D is embedded into K, namely

$$K = \left\{ \frac{r}{s} \mid r \in D, s \in D \backslash \{0\} \right\},$$

the set of all equivalence classes in $D \times D \backslash \{0\}$ built by the relation $(r_1, s_1) \sim (r_2, s_2)$ iff $s_2 r_1 = s_1 r_2$, i.e. $\frac{r_1}{s_1} = \frac{r_2}{s_2}$ iff $s_2 r_1 = s_1 r_2$.
(K is called **quotient field**/**field of quotients**/**field of fractions**.)

Theorem 2

There exists a K-vector space V such that M is embedded into V, namely

$$V = \left\{ \frac{m}{s} \mid m \in M, s \in D \backslash \{0\} \right\},$$

the set of all equivalence classes in $M \times D \backslash \{0\}$ built by the relation $(m_1, s_1) \sim (m_2, s_2)$ iff $s_2 m_1 = s_1 m_2$, i.e. $\frac{m_1}{s_1} = \frac{m_2}{s_2}$ iff $s_2 m_1 = s_1 m_2$.
Since $K \cdot M = V$, denote V as KM.
(KM is called **quotient module**/**module of quotients**/**module of fractions**).

To bring into modules, we need module version of these:

- For all $I \subseteq K$, let $I^{-1} := \{k \in K \mid kI \subseteq D\}$, then $I^{-1}I \subseteq D$.
- I is **invertible** if $I^{-1}I = D$.
- Let $I_v := (I^{-1})^{-1}$, then $I \subseteq I_v$.
- I is **v-ideal** if $I = I_v$.
- I is **v-invertible** if $(I^{-1}I)_v = D$.

Definitions

- For all $N \subseteq KM$, let $N^- := \{k \in K \mid kN \subseteq M\}$, then $N^-N \subseteq M$.[1]
- N is **invertible** if $N^-N = M$.[1]
- For all $I \subseteq K$, let $I^+ := \{x \in KM \mid Ix \subseteq M\}$, then $II^+ \subseteq M$.[2]
- Let $N_v := (N^-)^+$, then $N \subseteq N_v$.[2]
- N is **v-submodule**[2] if $N = N_v$.
- N is **v-invertible**[2] if $(N^-N)_v = M$.

[1] Naoum, A. G., Al-Alwan, F. H., 1996, *Dedekind Modules*, Communications in Algebra, Vol. 24, No. 2, p. 397-412
[2] Wijayanti, I. E., Marubayashi, H., Ernanto, I., Sutopo, 2020, *Finitely Generated Torsion-free Modules over Integrally Closed Domains*, Communications in Algebra, Vol. 48, Issue 8, p. 3597-3607

To bring into modules, we need module version of these:

- For all $I \subseteq K$, let $I^{-1} := \{k \in K \mid kI \subseteq D\}$, then $I^{-1}I \subseteq D$.
- I is **invertible** if $I^{-1}I = D$.
- Let $I_v := (I^{-1})^{-1}$, then $I \subseteq I_v$.
- I is **v-ideal** if $I = I_v$.
- I is **v-invertible** if $(I^{-1}I)_v = D$.

Definitions

- For all $N \subseteq KM$, let $N^{-} := \{k \in K \mid kN \subseteq M\}$, then $N^{-}N \subseteq M$.[1]
- N is **invertible** if $N^{-}N = M$.[1]
- For all $I \subseteq K$, let $I^{+} := \{x \in KM \mid Ix \subseteq M\}$, then $II^{+} \subseteq M$.[2]
- Let $N_v := (N^{-})^{+}$, then $N \subseteq N_v$.[2]
- N is **v-submodule**[2] if $N = N_v$.
- N is **v-invertible**[2] if $(N^{-}N)_v = M$.

[1] Naoum, A. G., Al-Alwan, F. H., 1996, *Dedekind Modules*, Communications in Algebra, Vol. 24, No. 2, p. 397-412

[2] Wijayanti, I. E., Marubayashi, H., Ernanto, I., Sutopo, 2020, *Finitely Generated Torsion-free Modules over Integrally Closed Domains*, Communications in Algebra, Vol. 48, Issue 8, p. 3597-3607

Definition 3

D is **Krull**[3] if

1. every v-ideals of D are v-invertible (i.e. D is cic),
2. D satisfies ascending chain condition on v-ideals.

Definition 4

Let M be finitely-generated. M is **Krull**[4] if

1. every v-submodules N of M satisfying $KN = KM$ are v-invertible,
2. M satisfies acc on v-submodules N_i where $KN_i = KM$.

Why $KN = KM$?

- $(\forall I \trianglelefteq D) KI = KD$
  $\Leftrightarrow$ field of fractions of I = field of fractions of D
- Take those N with the same module of fractions as M
  $\Leftrightarrow KN = KM$

[3] Gilmer, R., 1992, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Applied Mathematics, Vol. 90.

[4] Wijayanti, I. E., Marubayashi, H., Ernanto, I., Sutopo, 2022, *Arithmetic Modules over Generalized Dedekind Domains*, Journal of Algebra and Its Applications, Vol. 21, No. 03, 2250045.

Definition 3

D is **Krull**[3] if

1. every v-ideals of D are v-invertible (i.e. D is cic),
2. D satisfies ascending chain condition on v-ideals.

Definition 4

Let M be finitely-generated. M is **Krull**[4] if

1. every v-submodules N of M satisfying $KN = KM$ are v-invertible,
2. M satisfies acc on v-submodules N_i where $KN_i = KM$.

Why $KN = KM$?

- $(\forall I \trianglelefteq D) KI = KD$
  $\Leftrightarrow$ field of fractions of I = field of fractions of D
- Take those N with the same module of fractions as M
  $\Leftrightarrow$ $KN = KM$

---

[3] Gilmer, R., 1992, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Applied Mathematics, Vol. 90.

[4] Wijayanti, I. E., Marubayashi, H., Ernanto, I., Sutopo, 2022, *Arithmetic Modules over Generalized Dedekind Domains*, Journal of Algebra and Its Applications, Vol. 21, No. 03, 2250045.

Definition 3

D is **Krull**[3] if

1. every v-ideals of D are v-invertible (i.e. D is cic),
2. D satisfies ascending chain condition on v-ideals.

Definition 4

Let M be finitely-generated. M is **Krull**[4] if

1. every v-submodules N of M satisfying $KN = KM$ are v-invertible,
2. M satisfies acc on v-submodules N_i where $KN_i = KM$.

Why $KN = KM$?

- $(\forall I \trianglelefteq D) KI = KD$
  $\Leftrightarrow$ field of fractions of I = field of fractions of D
- Take those N with the same module of fractions as M
  $\Leftrightarrow$ $KN = KM$

[3] Gilmer, R., 1992, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Applied Mathematics, Vol. 90.

[4] Wijayanti, I. E., Marubayashi, H., Ernanto, I., Sutopo, 2022, *Arithmetic Modules over Generalized Dedekind Domains*, Journal of Algebra and Its Applications, Vol. 21, No. 03, 2250045.

Definition 3

D is **Krull**[3] if

1. every v-ideals of D are v-invertible (i.e. D is cic),
2. D satisfies ascending chain condition on v-ideals.

Definition 4

Let M be finitely-generated. M is **Krull**[4] if

1. every v-submodules N of M satisfying $KN = KM$ are v-invertible,
2. M satisfies acc on v-submodules N_i where $KN_i = KM$.

Why $KN = KM$?

- $(\forall I \trianglelefteq D) KI = KD$
  $\Leftrightarrow$ field of fractions of I = field of fractions of D
- Take those N with the same module of fractions as M
  $\Leftrightarrow$ $KN = KM$

[3] Gilmer, R., 1992, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Applied Mathematics, Vol. 90.

[4] Wijayanti, I. E., Marubayashi, H., Ernanto, I., Sutopo, 2022, *Arithmetic Modules over Generalized Dedekind Domains*, Journal of Algebra and Its Applications, Vol. 21, No. 03, 2250045.

Definition 4

Let M be finitely-generated. M is **Krull**[4] if

1. every v-submodule N of M satisfying $KN = KM$ is v-invertible,
2. M satisfies acc on v-submodules N_i where $KN_i = KM$.

We need to expand this to non-finitely-generated modules.

Problem: All v-submodules are v-invertible $\not\Rightarrow$ completely integrally closed

e.g. $M = \left\{ \frac{a}{2^n} \,\middle|\, a \in \mathbb{Z},\ n \in \mathbb{N}_0 \right\}$ as a $\mathbb{Z}$-module.

It'll be cic if $M^- = D$.

Definition 5

M is **Krull**[5] if

1. M is completely integrally closed modules (i.e. $M^- = D$ and every v-submodules N of M satisfying $KN = KM$ are v-invertible),
2. M satisfies acc on v-submodules N_i where $KN_i = KM$.

[4] Wijayanti, I. E., Marubayashi, H., Ernanto, I., Sutopo, 2022, *Arithmetic Modules over Generalized Dedekind Domains*, Journal of Algebra and Its Applications, Vol. 21, No. 03, 2250045.

[5] Nurwigantara, M. M., Wijayanti, I. E., Marubayashi, H., Wahyuni, S., *Krull Modules and Completely Integrally Closed Modules*, Journal of Algebra and Its Applications, Vol.21, No. 1, 2350038.

Definition 4

Let M be finitely-generated. M is **Krull**[4] if

1. every v-submodule N of M satisfying $KN = KM$ is v-invertible,
2. M satisfies acc on v-submodules N_i where $KN_i = KM$.

We need to expand this to non-finitely-generated modules.

Problem: All v-submodules are v-invertible $\not\Rightarrow$ completely integrally closed

e.g. $M = \left\{ \frac{a}{2^n} \,\middle|\, a \in \mathbb{Z},\ n \in \mathbb{N}_0 \right\}$ as a $\mathbb{Z}$-module.

It'll be cic if $M^- = D$.

Definition 5

M is **Krull**[5] if

1. M is completely integrally closed modules (i.e. $M^- = D$ and every v-submodules N of M satisfying $KN = KM$ are v-invertible),
2. M satisfies acc on v-submodules N_i where $KN_i = KM$.

[4] Wijayanti, I. E., Marubayashi, H., Ernanto, I., Sutopo, 2022, *Arithmetic Modules over Generalized Dedekind Domains*, Journal of Algebra and Its Applications, Vol. 21, No. 03, 2250045.

[5] Nurwigantara, M. M., Wijayanti, I. E., Marubayashi, H., Wahyuni, S., *Krull Modules and Completely Integrally Closed Modules*, Journal of Algebra and Its Applications, Vol.21, No. 1, 2350038.

Definition 4

Let M be finitely-generated. M is **Krull**[4] if

1. every v-submodule N of M satisfying $KN = KM$ is v-invertible,
2. M satisfies acc on v-submodules N_i where $KN_i = KM$.

We need to expand this to non-finitely-generated modules.

Problem: All v-submodules are v-invertible $\not\Rightarrow$ completely integrally closed

e.g. $M = \left\{ \frac{a}{2^n} \;\middle|\; a \in \mathbb{Z},\; n \in \mathbb{N}_0 \right\}$ as a $\mathbb{Z}$-module.

It'll be cic if $M^- = D$.

Definition 5

M is **Krull**[5] if

1. M is completely integrally closed modules (i.e. $M^- = D$ and every v-submodules N of M satisfying $KN = KM$ are v-invertible),
2. M satisfies acc on v-submodules N_i where $KN_i = KM$.

[4] Wijayanti, I. E., Marubayashi, H., Ernanto, I., Sutopo, 2022, *Arithmetic Modules over Generalized Dedekind Domains*, Journal of Algebra and Its Applications, Vol. 21, No. 03, 2250045.

[5] Nurwigantara, M. M., Wijayanti, I. E., Marubayashi, H., Wahyuni, S., *Krull Modules and Completely Integrally Closed Modules*, Journal of Algebra and Its Applications, Vol.21, No. 1, 2350038.

Let's go back to the $KN = KM$ issue.
Look at how Naoum and Al-Alwan defined Dedekind module as an analogy of Dedekind domain.

Definition 6

D is **Dedekind**[3] if every non-zero ideals of D are invertible.

Definition 7

M is **Dedekind**[1] if every non-zero submodules of M are invertible.

- Naoum and Al-Alwan doesn't need $KN = KM$.
- Many examples of M whose submodules N do not satisfy $KN = KM$, e.g. $M = R \oplus R \oplus R$, $N = R \oplus \{0\} \oplus R$

What if we generalize Krull domain in this way?

[3] Gilmer, R., 1992, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Applied Mathematics, Vol. 90.

[1] Naoum, A. G., Al-Alwan, F. H., 1996, *Dedekind Modules*, Communications in Algebra, Vol. 24, No. 2, p. 397-412.

Let's go back to the $KN = KM$ issue.
Look at how Naoum and Al-Alwan defined Dedekind module as an analogy of Dedekind domain.

Definition 6

D is **Dedekind**[3] if every non-zero ideals of D are invertible.

Definition 7

M is **Dedekind**[1] if every non-zero submodules of M are invertible.

- Naoum and Al-Alwan doesn't need $KN = KM$.
- Many examples of M whose submodules N do not satisfy $KN = KM$, e.g. $M = R \oplus R \oplus R$, $N = R \oplus \{0\} \oplus R$

What if we generalize Krull domain in this way?

[3] Gilmer, R., 1992, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Applied Mathematics, Vol. 90.

[1] Naoum, A. G., Al-Alwan, F. H., 1996, *Dedekind Modules*, Communications in Algebra, Vol. 24, No. 2, p. 397-412.

Definition 5

M is **Krull**[5] if

1. M is completely integrally closed modules (i.e. $M^- = D$ and every v-submodules N of M satisfying $KN = KM$ are v-invertible),
2. M satisfies acc on v-submodules N_i where $KN_i = KM$.

Definition 8

M is **strongly Krull**[8] if

1. M is strongly completely integrally closed modules (i.e. $M- = D$ and every v-submodules N of M are v-invertible),
2. M satisfies acc on v-submodules.

Note: This is similar to the one by Kim and Kim (2013)[7], but in Kim's case, they only see multiplication modules case.

Example 9

$\mathbb{Z} \times \mathbb{Z}[\frac{1}{2}]$ over $\mathbb{Z}$ is Krull but not strongly Krull

[5] Nurwigantara, M. M., Wijayanti, I. E., Marubayashi, H., Wahyuni, S., *Krull Modules and Completely Integrally Closed Modules*, Journal of Algebra and Its Applications, Vol.21, No. 1, 2350038.

[6] Nurwigantara, M. M., Wijayanti, I. E., Marubayashi, H., Wahyuni, S., *Krull Modules over Integral Domains*, Journal of Algebra and Its Applications, Vol. 24, No. 2, 2550061.

[7] Kim, H., Kim, M. O., 2013, *Krull Modules*, Algebra Colloquium, Vol. 20, No. 03, p. 463-474.

Definition 5

M is **Krull**[5] if

1. M is completely integrally closed modules (i.e. $M^- = D$ and every v-submodules N of M satisfying $KN = KM$ are v-invertible),
2. M satisfies acc on v-submodules N_i where $KN_i = KM$.

Definition 8

M is **strongly Krull**[8] if

1. M is strongly completely integrally closed modules (i.e. $M- = D$ and every v-submodules N of M are v-invertible),
2. M satisfies acc on v-submodules.

Note: This is similar to the one by Kim and Kim (2013)[7], but in Kim's case, they only see multiplication modules case.

Example 9

$\mathbb{Z} \times \mathbb{Z}[\frac{1}{2}]$ over $\mathbb{Z}$ is Krull but not strongly Krull

[5] Nurwigantara, M. M., Wijayanti, I. E., Marubayashi, H., Wahyuni, S., *Krull Modules and Completely Integrally Closed Modules*, Journal of Algebra and Its Applications, Vol.21, No. 1, 2350038.

[6] Nurwigantara, M. M., Wijayanti, I. E., Marubayashi, H., Wahyuni, S., *Krull Modules over Integral Domains*, Journal of Algebra and Its Applications, Vol. 24, No. 2, 2550061.

[7] Kim, H., Kim, M. O., 2013, *Krull Modules*, Algebra Colloquium, Vol. 20, No. 03, p. 463-474.

There's another generalization of Krull domain in module case by Costa and Johnson (1976).

Definitions 10

- For $m, m' \in M$, $m|m'$ if $(\exists r \in D)m' = rm$.
- m is **primitive** if $(\forall r \in D\backslash\{0\})(\forall m' \in M)(m|rm' \Rightarrow m|m')$.

Definition 11

M is **Krull**[8] in the sense of Costa-Johnson if

1. (KCJ1) $M = \bigcap_{\mathfrak{p} \in \min(D)} M_{\mathfrak{p}}$ where $\min(D) = \{\mathfrak{p} \trianglelefteq D \mid \mathfrak{p} \text{ minimal prime}\}$,
2. (KCJ2) $\forall \mathfrak{p} \in \min(D)$,
   1. $(\forall N' \trianglelefteq M_{\mathfrak{p}}\ v\text{-submodule})(\exists k \in D_{\mathfrak{p}})N' = kM_{\mathfrak{p}}$,
   2. $O_K(M_{\mathfrak{p}}) = D_{\mathfrak{p}}$,
   3. $M_{\mathfrak{p}}$ satisfies acc on v-submodules
3. (KCJ3) $(\forall m \in M\backslash\{0\})$ m is primitive in all but a finite number of $M_{\mathfrak{p}}$.[3]

[8] Costa, D. L. and Johnson, J. L., 1976, *Inert Extensions of Krull Domains*, Proceedings of the American Mathematical Society, Vol. 59, No. 2, p. 189-194.

There's another generalization of Krull domain in module case by Costa and Johnson (1976).

Definitions 10

- For $m, m' \in M$, $m|m'$ if $(\exists r \in D)m' = rm$.
- m is **primitive** if $(\forall r \in D\backslash\{0\})(\forall m' \in M)(m|rm' \Rightarrow m|m')$.

Definition 11

M is **Krull**[8] in the sense of Costa-Johnson if

1. (KCJ1) $M = \bigcap_{\mathfrak{p} \in \min(D)} M_{\mathfrak{p}}$ where $\min(D) = \{\mathfrak{p} \trianglelefteq D \mid \mathfrak{p} \text{ minimal prime}\}$,
2. (KCJ2) $\forall \mathfrak{p} \in \min(D)$,
   1. $(\forall N' \trianglelefteq M_{\mathfrak{p}}\ v\text{-submodule})(\exists k \in D_{\mathfrak{p}})N' = kM_{\mathfrak{p}}$,
   2. $O_K(M_{\mathfrak{p}}) = D_{\mathfrak{p}}$,
   3. $M_{\mathfrak{p}}$ satisfies acc on v-submodules
3. (KCJ3) $(\forall m \in M\backslash\{0\})$ m is primitive in all but a finite number of $M_{\mathfrak{p}}$.[3]

[8] Costa, D. L. and Johnson, J. L., 1976, *Inert Extensions of Krull Domains*, Proceedings of the American Mathematical Society, Vol. 59, No. 2, p. 189-194.

There's another generalization of Krull domain in module case by Costa and Johnson (1976).

Definitions 10

- For $m, m' \in M$, $m|m'$ if $(\exists r \in D)m' = rm$.
- m is **primitive** if $(\forall r \in D\backslash\{0\})(\forall m' \in M)(m|rm' \Rightarrow m|m')$.

Definition 11

M is **Krull**[8] in the sense of Costa-Johnson if

1. (KCJ1) $M = \bigcap_{\mathfrak{p}\in\min(D)} M_{\mathfrak{p}}$ where $\min(D) = \{\mathfrak{p} \trianglelefteq D \mid \mathfrak{p} \text{ minimal prime}\}$,
2. (KCJ2) $\forall \mathfrak{p} \in \min(D)$,
   1. $(\forall N' \trianglelefteq M_{\mathfrak{p}}\ v\text{-submodule})(\exists k \in D_{\mathfrak{p}})N' = kM_{\mathfrak{p}}$,
   2. $O_K(M_{\mathfrak{p}}) = D_{\mathfrak{p}}$,
   3. $M_{\mathfrak{p}}$ satisfies acc on v-submodules
3. (KCJ3) $(\forall m \in M\backslash\{0\})$ m is primitive in all but a finite number of $M_{\mathfrak{p}}$.[3]

[8] Costa, D. L. and Johnson, J. L., 1976, *Inert Extensions of Krull Domains*, Proceedings of the American Mathematical Society, Vol. 59, No. 2, p. 189-194.

Theorem 12

$$\left[\begin{array}{r} D \text{ Krull domain} \\ M \text{ Krull module Costa-Johnson} \end{array}\right] \Rightarrow M \text{ strongly Krull} \Rightarrow M \text{ Krull}$$

Sketch of Proof

1. $$\left[\begin{array}{c} KCJ2 \Rightarrow (\forall \mathfrak{p} \in \min(D)) M_{\mathfrak{p}} \text{ scic} \\ D \text{ Krull} \Rightarrow D = \bigcap_{\mathfrak{p} \in \min(D)} D_{\mathfrak{p}} \end{array}\right] \Rightarrow M \text{ scic}$$

2. KCJ2 $\Rightarrow$ m primitive in $M_{\mathfrak{p}}$ iff $m \in M \backslash \mathfrak{p} M_{\mathfrak{p}}$

   $N \neq 0 \Rightarrow N_{\mathfrak{p}} \nsubseteq \mathfrak{p} M_{\mathfrak{p}} \Rightarrow (N_{\mathfrak{p}})_v = M_{\mathfrak{p}}$ for all but finite number of $\mathfrak{p}$, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$

   $N = \bigcap_{\mathfrak{p} \in \min(D)} N_{\mathfrak{p}} \subseteq \mathfrak{p}^{l_1} M_{\mathfrak{p}_1} \cap \ldots \cap \mathfrak{p}^{l_n} M_{\mathfrak{p}_n} \cap \left(\bigcap_{\mathfrak{p} \in \min^*(D)} M_{\mathfrak{p}} \right)$ where $\min^*(D) = \min(D) \backslash \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$

   Put $\mathfrak{a} = \mathfrak{p}_1^{l_1} \cdots \mathfrak{p}_n^{l_n}$, then $N \subseteq (\mathfrak{a}M)_{\mathfrak{p}_1} \cap \ldots \cap (\mathfrak{a}M)_{\mathfrak{p}_n} \cap \left(\bigcap_{\mathfrak{p} \in \min^*(D)} M_{\mathfrak{p}} \right)$ $\subseteq (\mathfrak{a}_v M)_v \subseteq N$, thus $N = (\mathfrak{a}_v M)_v$.

Theorem 12

$$\left[\begin{array}{r} D \text{ Krull domain} \\ M \text{ Krull module Costa-Johnson} \end{array}\right] \Rightarrow M \text{ strongly Krull} \Rightarrow M \text{ Krull}$$

Sketch of Proof

1. $$\left[\begin{array}{c} KCJ2 \Rightarrow (\forall \mathfrak{p} \in \min(D)) M_{\mathfrak{p}} \text{ scic} \\ D \text{ Krull} \Rightarrow D = \bigcap_{\mathfrak{p} \in \min(D)} D_{\mathfrak{p}} \end{array}\right] \Rightarrow M \text{ scic}$$

2. KCJ2 $\Rightarrow$ m primitive in $M_{\mathfrak{p}}$ iff $m \in M \backslash \mathfrak{p} M_{\mathfrak{p}}$
   $N \neq 0 \Rightarrow N_{\mathfrak{p}} \not\subseteq \mathfrak{p} M_{\mathfrak{p}} \Rightarrow (N_{\mathfrak{p}})_v = M_{\mathfrak{p}}$ for all but finite number of $\mathfrak{p}$, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$

   $N = \bigcap_{\mathfrak{p} \in \min(D)} N_{\mathfrak{p}} \subseteq \mathfrak{p}^{l_1} M_{\mathfrak{p}_1} \cap \ldots \cap \mathfrak{p}^{l_n} M_{\mathfrak{p}_n} \cap \left(\bigcap_{\mathfrak{p} \in \min^*(D)} M_{\mathfrak{p}} \right)$ where $\min^*(D) = \min(D) \backslash \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$

   Put $\mathfrak{a} = \mathfrak{p}_1^{l_1} \cdots \mathfrak{p}_n^{l_n}$, then $N \subseteq (\mathfrak{a} M)_{\mathfrak{p}_1} \cap \ldots \cap (\mathfrak{a} M)_{\mathfrak{p}_n} \cap \left(\bigcap_{\mathfrak{p} \in \min^*(D)} M_{\mathfrak{p}} \right)$
   $\subseteq (\mathfrak{a}_v M)_v \subseteq N$, thus $N = (\mathfrak{a}_v M)_v$.

Theorem 12

$$\left[\begin{array}{r} D \text{ Krull domain} \\ M \text{ Krull module Costa-Johnson} \end{array}\right] \Rightarrow M \text{ strongly Krull} \Rightarrow M \text{ Krull}$$

Sketch of Proof

1. $$\left[\begin{array}{c} KCJ2 \Rightarrow (\forall \mathfrak{p} \in \min(D)) M_{\mathfrak{p}} \text{ scic} \\ D \text{ Krull} \Rightarrow D = \bigcap_{\mathfrak{p} \in \min(D)} D_{\mathfrak{p}} \end{array}\right] \Rightarrow M \text{ scic}$$

2. KCJ2 $\Rightarrow$ m primitive in $M_{\mathfrak{p}}$ iff $m \in M \backslash \mathfrak{p} M_{\mathfrak{p}}$

   $N \neq 0 \Rightarrow N_{\mathfrak{p}} \not\subseteq \mathfrak{p} M_{\mathfrak{p}} \Rightarrow (N_{\mathfrak{p}})_v = M_{\mathfrak{p}}$ for all but finite number of $\mathfrak{p}$, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$

   $$N = \bigcap_{\mathfrak{p} \in \min(D)} N_{\mathfrak{p}} \subseteq \mathfrak{p}^{l_1} M_{\mathfrak{p}_1} \cap \ldots \cap \mathfrak{p}^{l_n} M_{\mathfrak{p}_n} \cap \left(\bigcap_{\mathfrak{p} \in \min^*(D)} M_{\mathfrak{p}} \right) \text{ where}$$
   $\min^*(D) = \min(D) \backslash \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$

   Put $\mathfrak{a} = \mathfrak{p}_1^{l_1} \cdots \mathfrak{p}_n^{l_n}$, then $N \subseteq (\mathfrak{a}M)_{\mathfrak{p}_1} \cap \ldots \cap (\mathfrak{a}M)_{\mathfrak{p}_n} \cap \left(\bigcap_{\mathfrak{p} \in \min^*(D)} M_{\mathfrak{p}} \right)$
   $\subseteq (\mathfrak{a}_v M)_v \subseteq N$, thus $N = (\mathfrak{a}_v M)_v$.

Sketch of Proof of Theorem 12 (cont.)

3. $$\begin{cases} r(\forall N \leq M)(N_v = N \Rightarrow (\exists \mathfrak{a} \trianglelefteq D) N = (\mathfrak{a}_v M)_v) \\ (\forall \mathfrak{a}, \mathfrak{b} \trianglelefteq D)((\mathfrak{a}_v M)_v \subseteq (\mathfrak{b}_v M)_v \Rightarrow \mathfrak{a}_v \subseteq \mathfrak{b}_v) \\ \text{D satisfies acc on } v\text{-ideals} \end{cases}$$
$\Rightarrow M$ satisfies acc on v-submodules.

Examples and Properties

Example 13

$\langle 2, x\rangle$ over $\mathbb{Z}[x]$ is strongly Krull but not Krull in the sense of Costa-Johnson

Proposition 14

Projective modules over Krull domains are strongly Krull modules.

Sketch of Proof

Let M be a projective module $\Rightarrow F := M \oplus M_1$ is free for some M_1.

- Localize F into $F_\mathfrak{p} = M_\mathfrak{p} \oplus (M_1)_\mathfrak{p}$ which is scic $\Rightarrow M_\mathfrak{p}$ is scic $\Rightarrow M$ is scic.
- F is Krull in the sense of Costa and Johnson $\Rightarrow$ prove acc on v-submodules N of M by putting $L = N \oplus \{0\} \subseteq F$ and so $N = (\mathfrak{a}M)_v$ for some $\mathfrak{a} \trianglelefteq D$.

Examples and Properties

Example 13

$\langle 2, x\rangle$ over $\mathbb{Z}[x]$ is strongly Krull but not Krull in the sense of Costa-Johnson

Proposition 14

Projective modules over Krull domains are strongly Krull modules.

Sketch of Proof

Let M be a projective module $\Rightarrow F := M \oplus M_1$ is free for some M_1.

- Localize F into $F_{\mathfrak{p}} = M_{\mathfrak{p}} \oplus (M_1)_{\mathfrak{p}}$ which is scic $\Rightarrow M_{\mathfrak{p}}$ is scic $\Rightarrow M$ is scic.
- F is Krull in the sense of Costa and Johnson $\Rightarrow$ prove acc on v-submodules N of M by putting $L = N \oplus \{0\} \subseteq F$ and so $N = (\mathfrak{a}M)_v$ for some $\mathfrak{a} \trianglelefteq D$.

Examples and Properties

Theorem 15

If D is Krull, then

max v-ideal $\Leftrightarrow$ prime v-ideal $\Leftrightarrow$ min prime ideal

Theorem 16

If M is strongly Krull, then

max v-submodule $\Leftrightarrow$ prime v-submodule $\Rightarrow$ min prime submodule

Example 17

$\mathbb{Z} \times \mathbb{Z}$ over $\mathbb{Z}$ is strongly Krull, $\mathbb{Z} \times \{0\}$ is minimal prime but not v-ideal

- Costa, D. L., Johnson, J. L., 1976, *Inert Extensions of Krull Domains*, Proceedings of the American Mathematical Society, Vol. 59, no. 2, p. 189-194.
- Gilmer, R., 1992, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Applied Mathematics, Vol. 90.
- Kim, H., Kim, M. O., 2013, *Krull Modules*, Algebra Colloquium, Vol. 20, No. 03, p. 463-474.
- Moghaderi, J., Nekooei, R., 2010, *Valuation, Discrete Valuation, and Dedekind Modules*, International Electronic Journal of Algebra, Vol. 8, p. 18-29.
- Naoum, A. G., Al-Alwan, F. H., 1996, *Dedekind Modules*, Communications in Algebra, Vol. 24, No. 2, p. 397-412.
- Nurwigantara, M. M., Wijayanti, I. E., Marubayashi, H., Wahyuni, S., *Krull Modules and Completely Integrally Closed Modules*, Journal of Algebra and Its Applications, Vol.21, No. 1, 2350038.

- Nurwigantara, M. M., Wijayanti, I. E., Marubayashi, H., Wahyuni, S., *Krull Modules over Integral Domains*, Journal of Algebra and Its Applications, Vol. 24, No. 2, 2550061.
- Wijayanti, I. E., Marubayashi, H., Ernanto, I., Sutopo, 2020, *Finitely Generated Torsion-free Modules over Integrally Closed Domains*, Communications in Algebra, Vol. 48, Issue 8, p. 3597-3607.
- Wijayanti, I. E., Marubayashi, H., Ernanto, I., Sutopo, 2022, *Arithmetic Modules over Generalized Dedekind Domains*, Journal of Algebra and Its Applications, Vol. 21, No. 03, 2250045.