# Factorizations in free group algebras

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# Aims and tools

Factorizations in group algebras of free groups

Cohn cf. [2]: Group algebras of free group are firs (free ideal rings), i.e., rings such that one-sided ideals are free of unique rank; elements of firs admit irreducible factorizations.

Aims: Find irreducible factorizations in free group algebras
Uniqueness, algorithm?

Tools: module types, link modules, lattices and localizations.

# W. Leavitt fundamental works

In 1955/1960 Leavitt asked and answered the question

*when does a finitely generated free module have a unique rank?*

via a notion of *module type*. Leavitt's works were well-accepted, highly esteemed by Cohn, Bergman and Corner... but a majority did realized neither their beauty nor their importance

Leavitt was ahead of his time for decades: Gröbner bases, flat epimorphisms were invented 15 years later in view of the fact that his proof was a marvelous use of the nowadays so-called Gröbner bases or more precisely, Schreier bases.

# Module type



**Theorem 1**

*Free R-modules are either pairwise non-isomorphic or* $\exists$ *smallest integers* $m < n$: $R^m \cong R^n$ *according to the* **module type** *of R is either* **IBN** *or* $(\mathbf{m}, \mathbf{n})$. *In the latter case, all* $R^l$ $(l < m)$ *have a unique rank l and* $\forall\ m \leq t \in \mathbb{N}\ \exists!\ m \leq s < n$: $R^t \cong R^s$.

IBN-rings cover ones with chain conditions or factors into skew fields etc. Module type $(1, 2)$ is obvious. Module type $(m, n)$: Algebraic theory of infinity no finiteness condition.

Leavitt 1955-1962: a construction of generic algebras of module type $(m, n)$ over $\mathbb{F}_2$ for $m = 1$, over $\mathbb{Z}, \mathbb{Q}$ for $(m, m+1), m \neq 1$.

Leavitt's *normal form*, i.e., "Gröbner's bases" though it was invented a decade later by Buchberger for polynomial algebras.

# Leavitt algebras and localizations I

1. Construction of a module type $(m, n)$ $1 \leq m \leq n - 1$.

$L_K(m, n)$ is a localization of a free algebra $A$ on $\{x_{ij}\}$ over a field $K$ by inverting the $m \times n$ matrix $(x_{ij})$.

2. Module type $(1, n)$ is well-understood but not $(m, n), m > 1$.

3. Computing module type is essentially one of a Grothendieck group.

4. Inverting a set $\Sigma$ of matrices. If matrices are full, square, it is a Cohn's localization. If $\Sigma$ is a row, then Leavitt's localization. Invertibility of matrices in $\Sigma$ is not enough to a universality even in the case of injectivity.

5. A matrix is full if it is not a product of non-square ones.

6. For a universal localization $R_\Sigma$ of $R$ by $\Sigma$ $\exists\, \jmath\colon R \to R_\Sigma$.
$I = \ker \jmath \Longrightarrow 1 + (a_{ij}) \in M_l(R_\Sigma), a_{ij} \in I, l \in \mathbb{N}$ invertible.

7. Jacobson's localization if $\Sigma = \{1 + M_l(I) \mid I \lhd R; l \in \mathbb{N}\}$.
If $I \subseteq J(R) \Longrightarrow R = R_\Sigma$.

8. By uniqueness of inverses, $\jmath\colon R \to R_\Sigma$ is a ring epimorphism.

9. Power series rings, inverse limits, transfinite nilpotence at Jacobson's localization.

10. Find handy constructions of $R_\Sigma$ for good $\Sigma$.

11. An old twin, the algebra of Fox derivatives is implicitly invented in 1953.

$L(R)$ Leavitt localization by $(x_1, \cdots, x_n)$ over $R$.

**Theorem 2**

*If $\jmath\colon R \to L(R)$ is injective, then $I_R = \sum x_i R = \oplus x_i R$ is free of rank $n$, $rl = 0 \Rightarrow r = 0$. If $I_R = \sum_{i=1}^{n} x_i R \lhd R \Rightarrow L(R) \cong Q(R)$ $= \varinjlim_{l \in \mathbb{N}} \mathrm{Hom}_R(I^l, R)$, a ring of right quotients of $R$.*

$x_i^\star \in \mathrm{Hom}_R(I, R_R)\colon x_i^\star(x_j) \mapsto \delta_{ij} = \begin{cases} 1 & \text{if} \quad j = i, \\ 0 & \text{if} \quad j \neq i \end{cases}$, then

over $Q(R)\colon (x_1, \cdots, x_n)^{-1} = \begin{pmatrix} x_1^\star \\ \vdots \\ x_n^\star \end{pmatrix}$.

# Jacobson localization

$I \lhd R$ *transfinitely nilpotent* $\Leftrightarrow \cap_{m=1}^{\infty} I^m = 0$. $\epsilon \colon R \to R/I = S$ the augmentation; $R \subseteq \varprojlim R/I^l = \hat{R}$; $I \subseteq J(\hat{R})$, the Jacobson radical. The *rational closure* $R_I^{\mathrm{rat}}$ of $R$ wrt $I$ is the smallest subalgebra of $\hat{R}$ containing $R$, quasi-inverses of its quasiregular elements (as an elements of $\hat{R}$).

## Theorem 3

*If $I \lhd R$ is transf. nilpotent, then as either a left or a right $R$-module, $R_I^{\mathrm{rat}}$ is generated by entries of inverses of square matrices $1 + Q$, respectively, where $Q$ runs over all square matrices whose entries belong to $I$ whence every matrix $1 + Q\,(Q \in M_I)$ is invertible.*

$I \lhd R; \Sigma = \{1 + M_l(I) \mid l \in \mathbb{N}\}$.

When is a universal Cohn localization $C_I(R)$ of $R$ by $\Sigma$ a Jacobson one?

$I$ must be necessarily transf. nilpotent.

# Basic examples

1. $A = K\langle x_1, \cdots, x_n\rangle$ free algebra over a commutative ring $K$.

2. $\Lambda = KF_n$ group algebra of a free group $F_n$.

3. $\Gamma$ non-commutative power series algebra over $K \supseteq \Lambda \supseteq A$.
$\epsilon\colon \Gamma \to K$ augmentation.
The ideal generated by $x_1, \cdots, x_n$ free of rank $n$.
$\Lambda = K\langle x_1, \cdots, x_n; y_1, \cdots, y_n\rangle / \langle x_i y_i - y_i x_i, 1 - x_1 y_i\rangle$.
$A_I^{\mathrm{rat}} = \Lambda_I^{\mathrm{rat}}$, $I$ augmentation ideal.

4. Fox calculus, geometric topology concern $\Lambda, \Gamma$ establishing connection to module type and Leavitt localization.

# Link modules I

$I_R \lhd R, rI = 0 \Rightarrow r = 0$ free on $x_1, \cdots, x_n$. $\epsilon \colon R \to R/I = S$.

$$0 \to {}^nR \xrightarrow{(x_1 \cdots x_n)} R \xrightarrow{\epsilon} S \to 0 \colon (x_1 \cdots x_n) \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = \sum_{i=1}^{n} x_i r_i$$

## Definition 4

${}_RM$ *weak link (Sato) module* wrt $I$ if $\mathrm{Tor}_*^R(S, M) = 0$, or equivalently, $\exists\, M^n \cong M$ such that

$$(1) \qquad (m_1, \cdots, m_n) \in M^n \longmapsto \sum_{i=1}^{n} x_i m_i = m \in M.$$

If ${}_RM$ is finitely presented, then $M$ *Sato module L-module*.

# Link modules II

1. Link modules is exactly left modules over Leavitt localization.
2. Assignments $m \mapsto m_i$ are (generalized) Fox derivations $\partial_i$'s

**Warning.** No symmetry: left modules but right localization!

3. New aspects: No symmetry as for Leavitt path algebras.
4. Applications of Leavitt localizations to geometric topology.
5. **Basic Examples for link modules**

5.1. $r \in R, \epsilon(r) = 1 \Rightarrow L(R)/L(R)r \cong R/Rr$ link modules.

5.2. $\rho \in 1 + M_I(I) \Rightarrow M = \text{coker}\, \rho_R = R^I/R^I \rho_R$ link module.

5.3. $\rho \in 1 + M_I(I)$, $\rho^{-1} = (b_{ij}) \in M_I(R_I^{\mathrm{rat}}) \Longrightarrow \sum_{i,j=1}^{I} Rb_{ij}/R$ link module $\Longrightarrow R_I^{\mathrm{rat}}/R$ weak link module.

# Link modules III

**Theorem 5**

*Let $S$ be a domain over a principal ideal domain $K$ such that $S$ is a free $K$-module of finite rank, and $R = S\langle x_1, \cdots, x_n\rangle$ a free algebra on $\{x_1, \cdots, x_n\}$. Then $R^{\mathrm{rat}}$ wrt $S\langle\langle x_1, \cdots, x_n\rangle\rangle$ is the universal Cohn localization of $R$ wrt $1 + M_I$ where $I = \sum_{i=1}^{n} x_i R$.*

## Some comments

There are two ways to compute $\mathrm{Tor}_*^R(I, M)$.
By a free resolution of $I$: $\mathrm{Tor}_*^R(I, M) = 0 \iff M$ link module
By a free resolution of a link module $M$ to a deeper study of $M$.
New aspects, methods to module theory of Leavitt localizations.

In contrast to the commutative case it is a difficult result of Cohn that group algebras of free groups are UFD. The best proof is due to Rosenmann-Rosset using Schreier bases.

**Cohn's example: [5, Exercise 5.3.5]**

$$xyzyx + xyz + zyx + xyx + x + z = (xyz + x + z)(yx + 1) =$$

$$= (xy + 1)(zyx + x + z)$$

$\Lambda = KF_n$ the group algebra of a free group $F_n$ of rank $n$ over a filed $K$. $L(\Lambda)$ the Leavitt localization of $\Lambda$. $\gamma \in \Lambda$ *comonic* $\iff \epsilon(\gamma) = 1$.

**Theorem 6**

*For any two polynomials $\gamma, \lambda \in \Lambda$ there is a uniquely determined (up to a unit) comonic polynomial $\delta \in \Lambda$, called, a greatest common divisor of $\gamma, \lambda$ by the property that $\delta$ is a generator of the left ideal of $L(\Lambda)$ generated by $\gamma$ and $\lambda$, that is, $L(\Lambda)\gamma + L(\Lambda)\lambda = L(\Lambda)\delta$*

**Theorem 7**

*Let $\pi \in \Lambda$ be a comonic polynomial which is not a unit. Then $\pi$ is irreducible if and only if the Sato module $\Lambda/\Lambda\pi$ is a simple module over the Leavitt localization $L(\Lambda)$.*

The main result

**Theorem 8**

*$\alpha \in \Lambda$ non-unital comonic; ${}_{L(\Lambda)}M = \Lambda/\Lambda\alpha$. Every irreducible factorization $\alpha = \rho_1 \cdot \rho_l$ corresponds to a composition chain*

$$0 \subseteq \frac{\Lambda\rho_2 \cdots \rho_l}{\Lambda\alpha} \subseteq \frac{\Lambda\rho_3 \cdots \rho_l}{\Lambda\alpha} \subseteq \cdots \subseteq \frac{\Lambda\rho_l}{\Lambda\alpha} \subseteq \frac{\Lambda}{\Lambda\alpha}$$

*of $M$. Each simple subfactor $\frac{\Lambda\rho_{j-1}\cdots\rho_l}{\Lambda\rho_j\cdots\rho_l}$ or $\frac{\Lambda}{\Lambda\rho_l}$ determines $\rho_j$ or $\rho_l$, respectively, only up to the similarity.*

$a, b \in R$ *similar* $\iff$ left (right) modules $R/Ra, R/Rb\,(R/aR, R/bR)$ isomorphic.

$A$ has a degree function ensuring irreducible factorizations.
$\Lambda$ has only an order function no degree function.
An order function is not suitable to studying factorizations.
$\Lambda$ has a length function suitable to studying factorizations.
A length function reduces the process by quite slowly.

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**Thank you for your attention!**