Factorization theory in (generalized) cluster algebras

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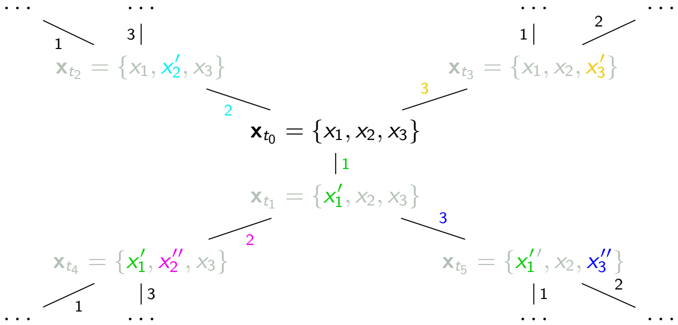
Cluster algebras are a class of commutative rings with extra combinatorial structure.

- A cluster algebra is a subalgebra of a rational field generated by a (possibly infinite) set of cluster variables.
- It begins with an initial set of algebraically independent variables over a field K, which are replaced using specific rules to generate new variables.
- The cluster variables are grouped in special subsets called clusters.
- Despite their dynamic nature, all generated variables are Laurent polynomials in the initial variables.

The Mutation Process



- Two adjacent clusters are related by $x_{i,t}x_{i,t'} = f_{i,t}$, where $f_{i,t}$ is a binomial in $\mathbf{x}_t$, called exchange polynomial.
- The exchange relations are given by a matrix B.
- The cluster algebra associated to B is the K-algebra generated by all possible cluster variables.

Generalized Cluster Algebras

- In a cluster algebra, the exchange polynomials are always of the form

$$f_{i,t} = \mathbf{x}_t^{\underline{\mathbf{m}}_t} + \mathbf{x}_t^{\underline{\mathbf{n}}_t},$$

with $\underline{\mathbf{m}}_t, \underline{\mathbf{n}}_t \in \mathbb{N}_0^n$ such that $m_j n_j = 0$.

- We can allow them to be of the form

$$f_{i,t} = \mathbf{x}_t^{d\underline{\mathbf{m}}_t} + \rho_{d-1}\mathbf{x}_t^{(d-1)\underline{\mathbf{m}}_t}\mathbf{x}_t^{\underline{\mathbf{n}}_t} + \cdots + \rho_1\mathbf{x}_t^{\underline{\mathbf{m}}_t}\mathbf{x}_t^{(d-1)\underline{\mathbf{n}}_t} + \mathbf{x}_t^{d\underline{\mathbf{n}}_t}.$$

- In this case we obtain the more general concept of generalized cluster algebra.

An Example

Let $\mathbf{x}_{t_0} = \{x_1, x_2\}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

$$\begin{array}{ccccc}
\mathbf{x}_{t_1} = \left\{\frac{1+x_2}{x_1}, x_2\right\} & \overset{1}{\text{---}} & \mathbf{x}_{t_0} = \{x_1, x_2\} & \overset{2}{\text{---}} & \mathbf{x}_{t_2} = \left\{x_1, \frac{x_1+1}{x_2}\right\} \\
\Big| \, 2 & & & & \Big| \, 1 \\
\mathbf{x}_{t_3} = \left\{\frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1 x_2}\right\} & & & & \mathbf{x}_{t_4} = \left\{\frac{1+x_1+x_2}{x_1 x_2}, \frac{x_1+1}{x_2}\right\}
\end{array}$$

$$A = K\left[x_1, x_2, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1 x_2}\right]$$

The History of Factorization Theory of Cluster Algebras

- 2002 — Fomin and Zelevinski: cluster algebras
- 2012 — Geiss, Leclerc, Schrörer: units and atoms
- 2012 — Lampe: factoriality of cluster algebras of type A, D, E
- 2014 — Chapiro and Checov: generalized cluster algebras
- 2019 — Garcia Elsener, Lampe, Smertnig: class group of an acyclic cluster algebra
- 2023 — Cao, Keller, Qin: factoriality of full rank upper cluster algebras
- 2025 — P. : class groups of full rank upper cluster algebras
- 2025 — P. : realization theorem of f.g. groups as class group of g.c.a.

GOAL: Let R be an integral domain. We want to factor non-invertible elements of R into irreducibles.[1]

- Krull domains are an important class of rings to which we can associate an invariant — the class group — that helps understand factorizations better.
- An integral domain R is UFD if and only if R is Krull with trivial class group.
- The class group of R is defined as $\mathcal{C}(R) = \langle \mathfrak{X}(R) \rangle / \mathcal{H}(R)$, where $\mathfrak{X}(R)$ denotes the set of height-one prime ideals of R, and $\mathcal{H}(R)$ is the set of its principal fractional ideals.

Many interesting classes of cluster algebras turn out to be Krull domains, i.e. acyclic cluster algebras, full rank upper cluster algebras, locally acyclic cluster algebras,...

[1] $u \in A$ is irreducible if $u = ab$, $a, b \in R$ implies that one between a and b is invertible.

Theorem (Garcia Elsener, Lampe, Smertnig, 2019)

Let A be a cluster algebra, and let $\{x_1, \dots, x_n\}$ be a cluster. Assume that A is a Krull domain. The class group $\mathcal{C}(A)$ of A is isomorphic to $\mathbb{Z}^r$, for some $r \geq 0$, and it is generated by the classes of the height-1 primes containing at least one initial cluster variable. Moreover, each class contains infinitely many divisors.

The rank can be computed explicitly (in terms of the initial data) for some classes of cluster algebras:

- When A is an acyclic cluster algebra. (Garcia Elsener, Lampe, Smertnig, 2019).
- When A is a full rank upper cluster algebra (P., 2025).

Realization Theorem

Theorem (P. 2025)

Let G be a finitely generated abelian group. Then there exists a generalized cluster algebra A such that the class group $\mathcal{C}(A)$ of A is isomorphic to G.

The Goal

Factorizations and class groups can be computed explicitly only in a few classes of rings, e.g.

- algorithms for the factorization of an integer into primes;
- algorithms for the factorization of polynomials;
- algorithms to compute the class group of a number field.

> We have developed a computational method to determine the class group and the factorizations of elements in a "reasonable class of commutative rings".

More specifically, we want a ring R such that:

- The height-1 spectrum of R is covered by finitely many height-1 spectra of Laurent polynomial rings.
- There are transformation maps between the field of fractions of each Laurent polynomial ring.

In case of cluster algebras, these maps are given by mutation sequences.

Our Motivating Example: Locally Acyclic Cluster Algebras

- Locally acyclic cluster algebras form a subclass of cluster algebras that are noetherian and integrally closed (Muller, 2013).
- They naturally arise from triangulations of surfaces.
- These algebras are Krull domains, yet no explicit formula for the rank of their class group was previously known.
- Every locally acyclic cluster algebra is an intersection of finitely many acyclic cluster algebras.
- Their height-1 spectrum is covered by finitely height-1 spectra of Laurent polynomial rings.

Theorem (P., Smertnig 202?)

Let A be a locally acyclic cluster algebra[a]. Then there exists an algorithm that computes the class group of A. Moreover, for any element $f \in A$, the algorithm can determine all factorizations of f into irreducible elements.

[a] The right class to consider is Banff algebra.

In practical terms, you input a matrix B defining the cluster algebra, and a Laurent polynomial f in the initial cluster variables. The algorithm returns:

- the rank of the class group of $A(B)$
- a list of irreducible divisors of f
- all factorizations of f expressed via their exponent vectors.

Termination requires B to have the right form. Matrices coming from triangulations of surfaces are valid examples.

Let $B = \begin{pmatrix} 0 & -1 & 0 & 4 \\ 2 & 0 & 3 & 6 \\ 0 & -3 & 0 & 0 \\ -4 & -3 & 0 & 0 \end{pmatrix}$, $K = \mathbb{C}$, $A = A(B)$ and $f = x_2^3 x_4^4 + x_2^5 + x_4^4 + x_2^2$

Then f has 4 different factorizations into irreducibles given by:

$$\begin{aligned} f &= (x_2^2 - x_2 + 1)(x_4^2 + ix_2)(x_4^2 + ix_2)(x_2 + 1) \\ &= x_1(x_2^2 - x_2 + 1)\left(\frac{x_4^4 + x_2^2}{x_0}\right)(x_2 + 1) \\ &= x_3(x_4^2 + ix_2)(x_4^2 + ix_2)\left(\frac{x_2^3 + 1}{x_3}\right) \\ &= x_1 x_3\left(\frac{x_2^3 + 1}{x_3}\right)\left(\frac{x_4^4 + x_2^2}{x_0}\right) \end{aligned}$$

Proposition (Muller 2013)

Every locally acyclic cluster algebra is finitely presented.

- The class group can be computed from a finite presentation.
- An algorithm for computing presentations of (some) locally acyclic cluster algebras has already been implemented (Maltherne, Muller, 2013).
- However, factorizations using this approach are computationally very expensive.
- To obtain a finite presentation, the algorithm uses Gröbner bases, which can have double exponential complexity.
- Our approach does not require Gröbner bases; it relies only on factorizations in multivariate polynomial rings.

Comparison of Algorithms

Runtime
for cluster algebras of type A

Vars	MM13	PS25
2	0.3s	0.07s
4	0.4s	0.2s
6	7.3s	0.4s
8	$>>$	0.6s
50	$>>$	18.7s

Summary of Key Differences

- MM13 fails beyond 6 variables.
- PS25 supports a wider class of locally acyclic cluster algebras.

Thank you for your attention!