July 14, 2025

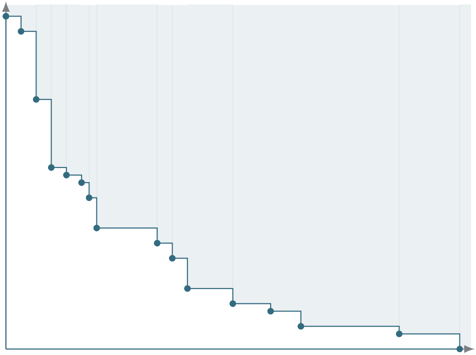
Picturing minimal generating sets of large powers of bivariate monomial ideals

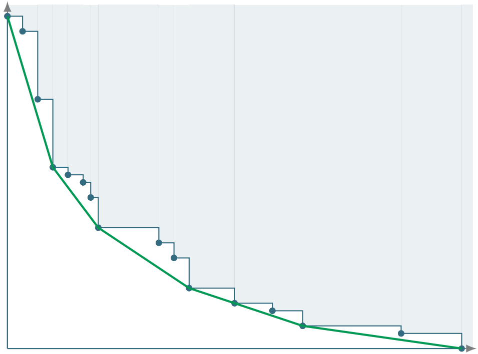
FWF Der Wissenschaftsfonds.

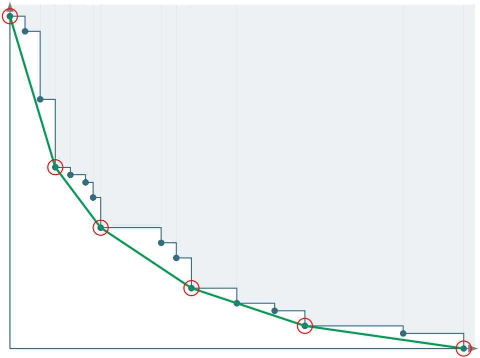
UNIVERSITÄT KLAGENFURT

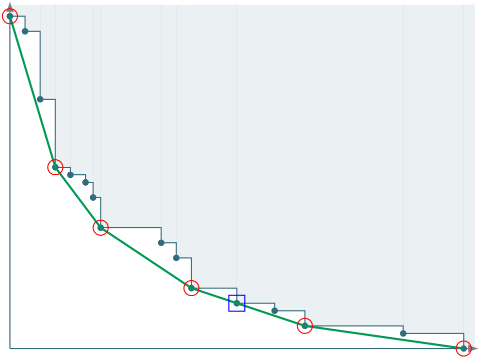
Modeling - Analysis - Optimization DOC.FUNDS DOCTORAL SCHOOL

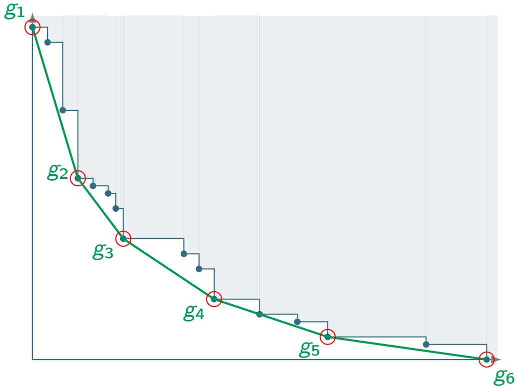
Roswitha Rissner



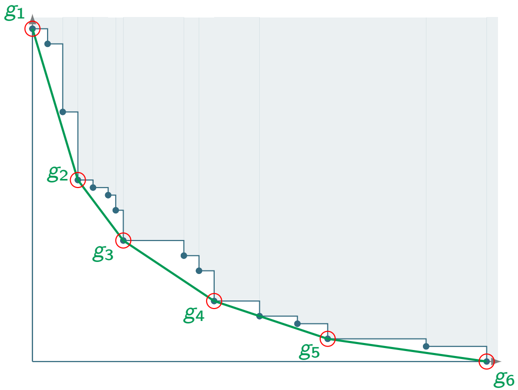






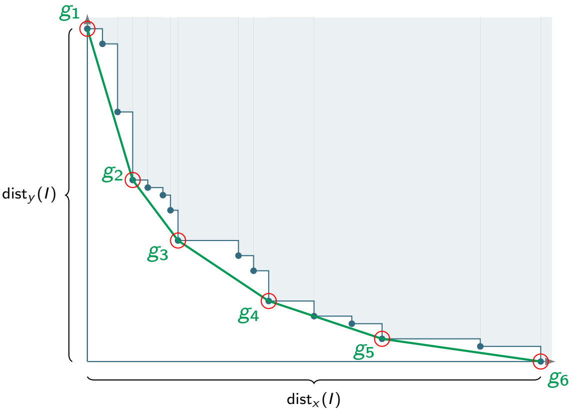


g_1
g_2
g_3
g_4
g_5
g_6



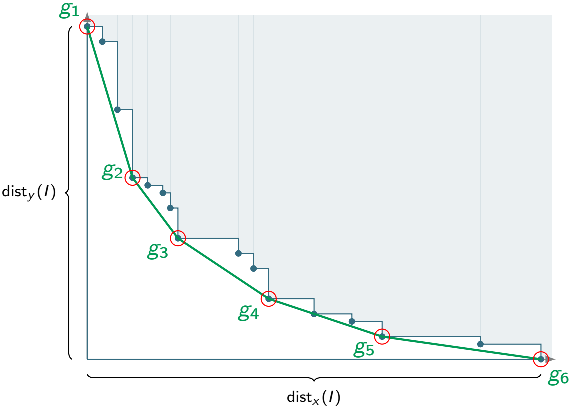
Theorem (Rath, R., 2025)

$$\forall \ell \geq 0\colon I^{D+\ell} = \sum_{i=1}^{k} (g_i, g_{i+1})^{\ell} I^{D}$$



Theorem (Rath, R., 2025)

$$\forall \ell \geq 0\colon\ I^{D+\ell} = \sum_{i=1}^{k} (g_i, g_{i+1})^{\ell} I^{D} \quad \text{with } D = \mu(I)\,\text{dist}_{\bullet}(I)^2$$



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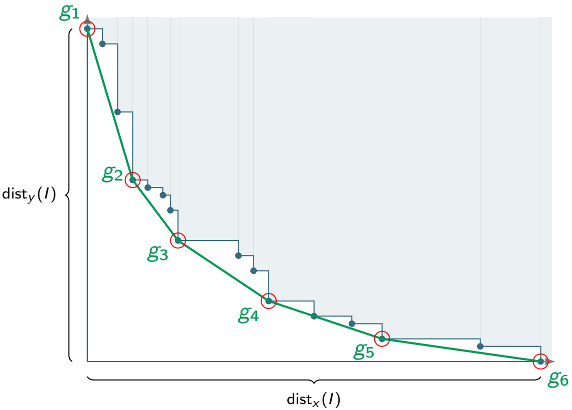

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One summand: $(g_i, g_{i+1})^{D+\ell} I^D$

One summand: $(g_i, g_{i+1})^{\overset{k}{\cancel{D+\ell}}} I^D$

One summand: $(g_i, g_{i+1})^{\overset{k}{\cancel{D+\ell}}} / \cancel{D}\ J$

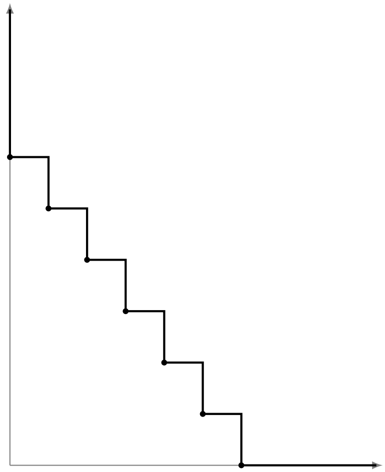
One summand: $(g_i, g_{i+1})^{\cancel{D+\ell}\, k} \cancel{I^D}\, J$

$= \gcd(g_i, g_{i+1})^k (x^u, y^v)^k J$

One summand: $(g_i, g_{i+1})^{\cancel{D+\ell}\,k} \cancel{I^D}\, J$

$= \gcd(g_i, g_{i+1})^k \boxed{(x^u, y^v)^k J}$

One summand $(x^u, y^v)^k J$



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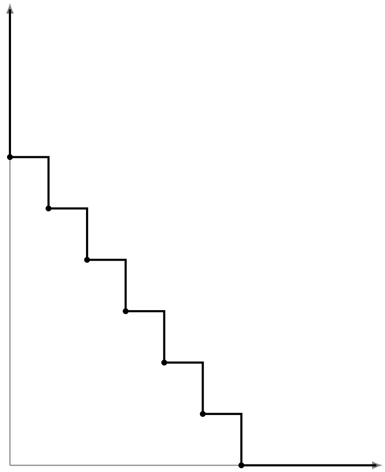

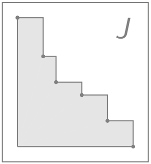

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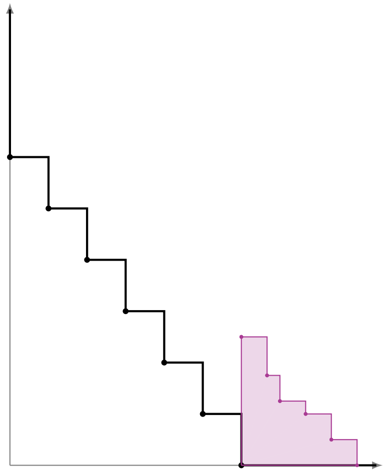

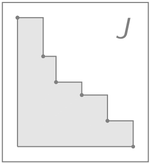

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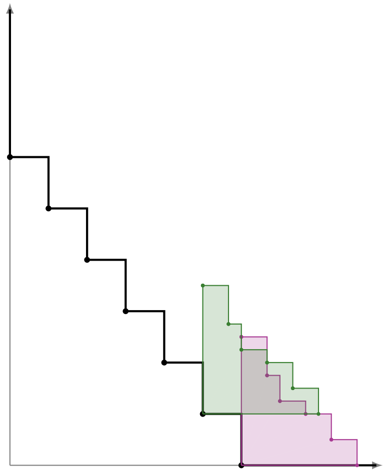

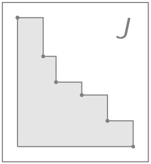

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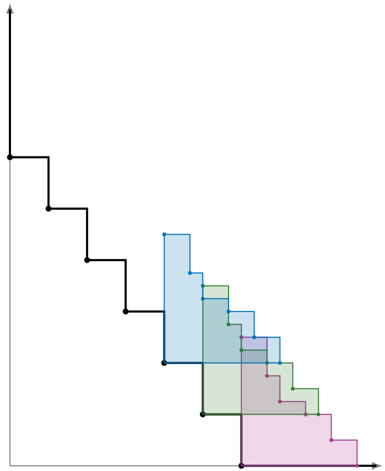

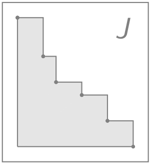

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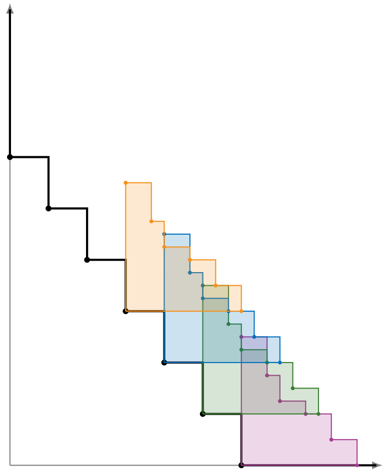

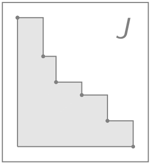

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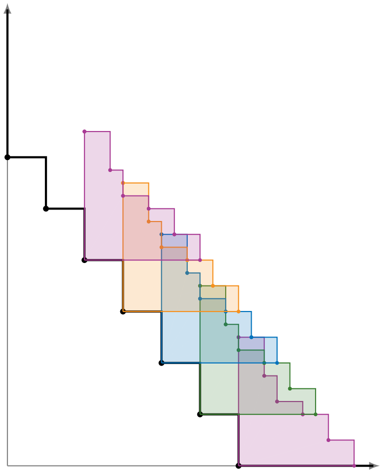

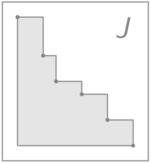

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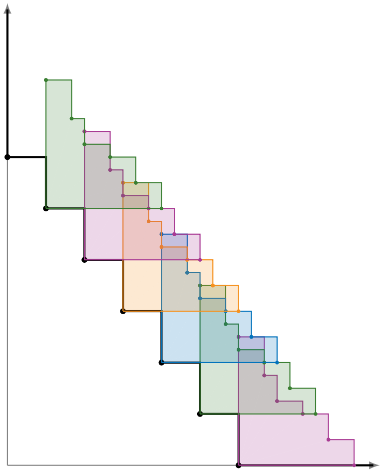

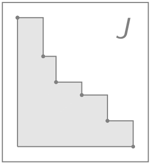

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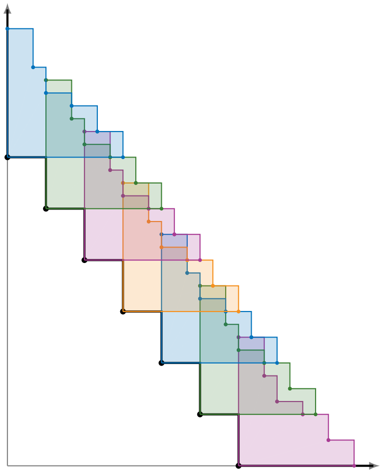

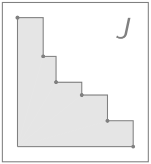

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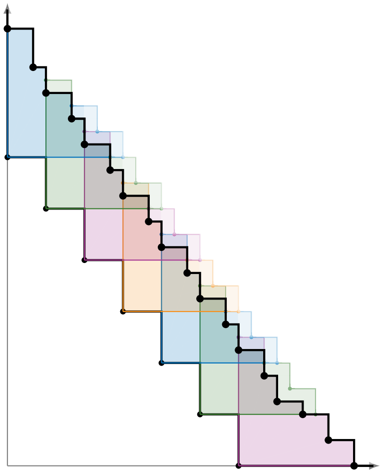

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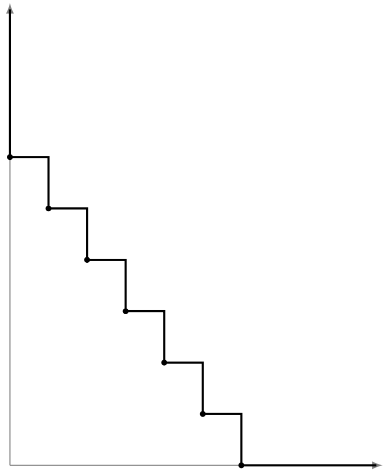

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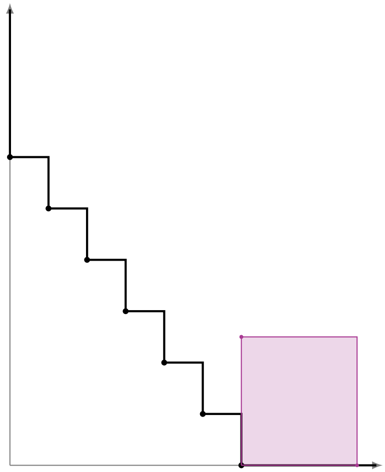
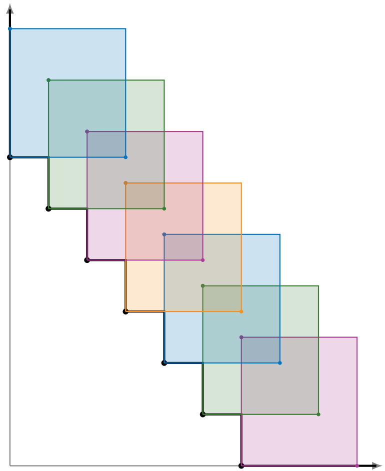


One summand $(x^u, y^v)^k J$



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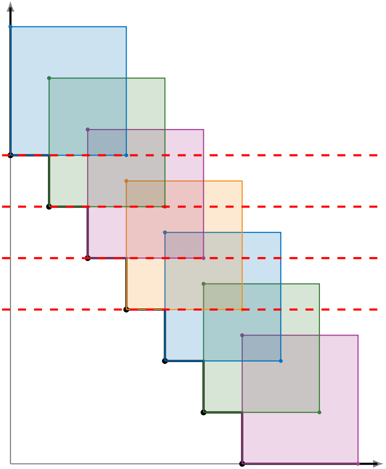

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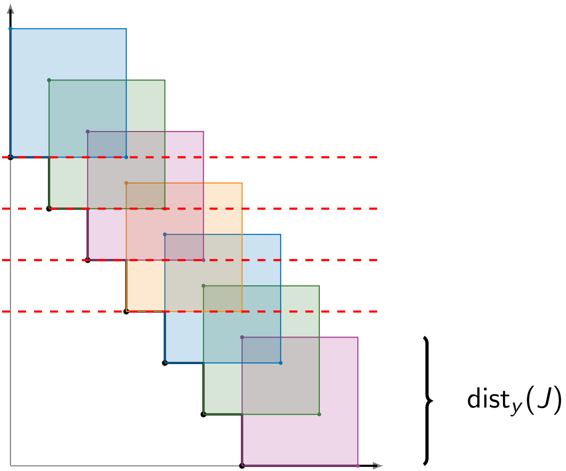

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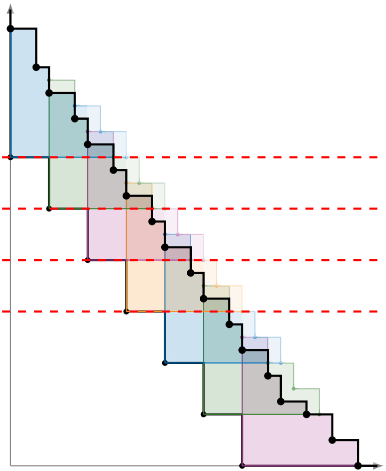

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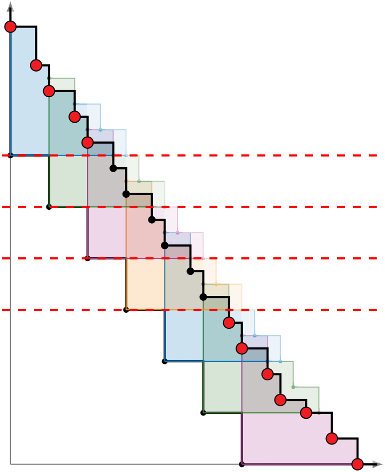

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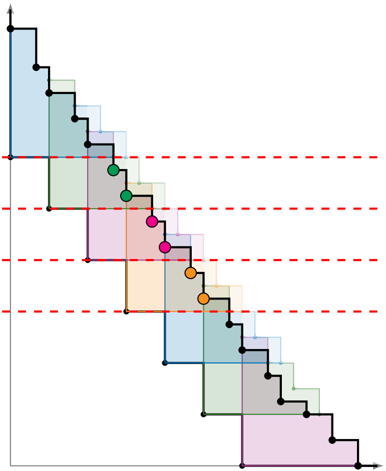

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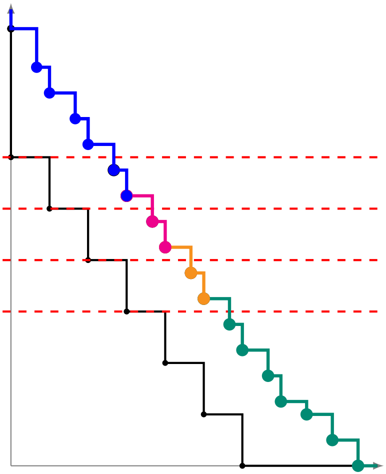

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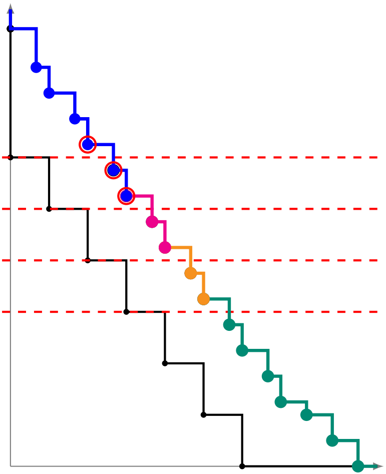
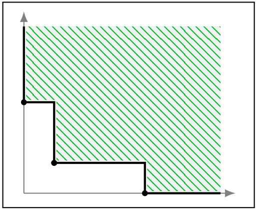
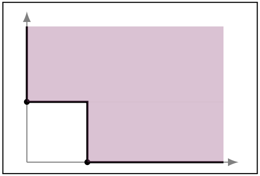


linking staircases $A \textcircled{y} B$

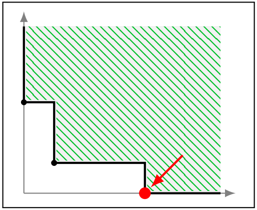


$A = (x^4, xy, y^3)$



$B = (x^2, y^2)$

linking staircases $A \textcircled{y} B$



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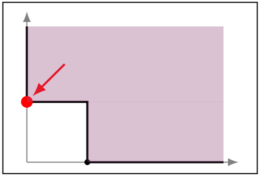

$B = (x^2, y^2)$

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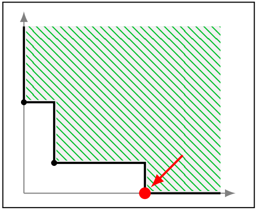

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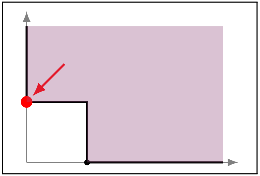

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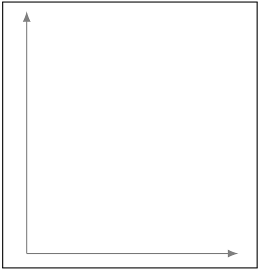
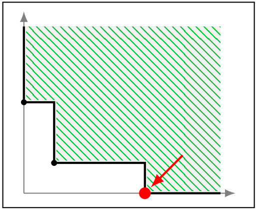


linking staircases $A \circledcirc_y B$



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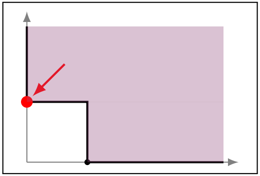

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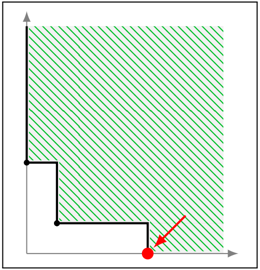
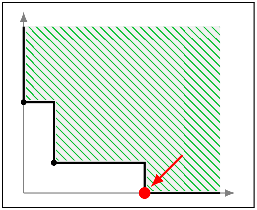
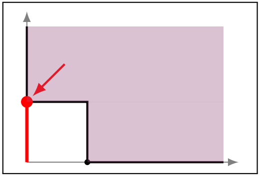


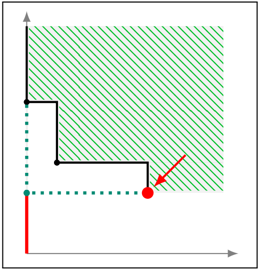
linking staircases $A \circledcirc_y B$



$A = (x^4, xy, y^3)$

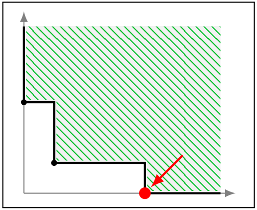


$B = (x^2, y^2)$

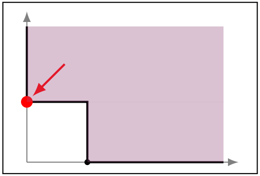


$y^2 A$

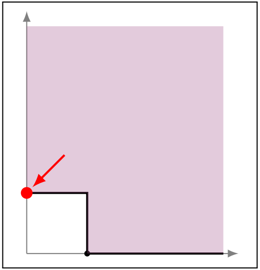
linking staircases $A \circledcirc_y B$



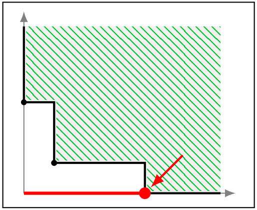
$A = (x^4, xy, y^3)$



$B = (x^2, y^2)$



linking staircases $A \textcircled{y} B$



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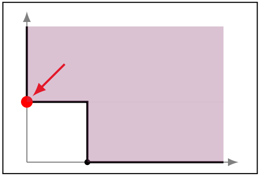

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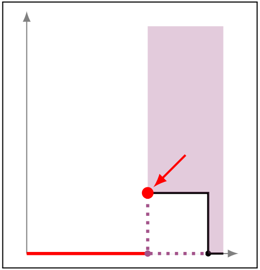
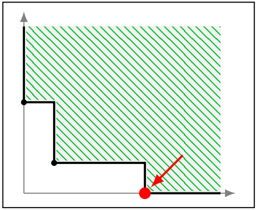


$x^4 B$

linking staircases $A \circledcirc_y B$



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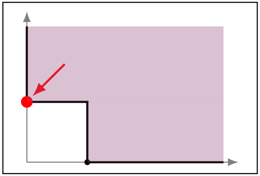
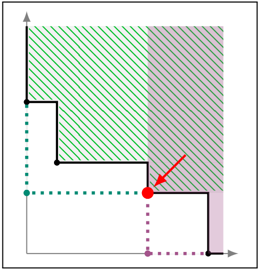
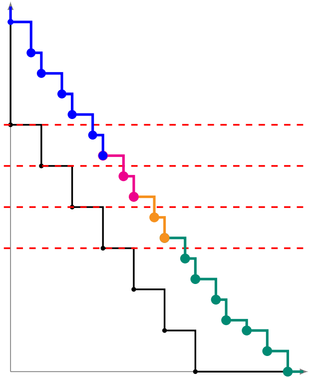


$B = (x^2, y^2)$



$A \circledcirc_y B = y^2 A + x^4 B$

One summand $(x^u, y^v)^k J$



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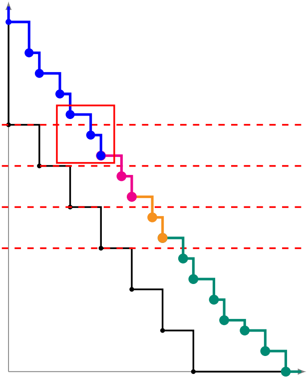

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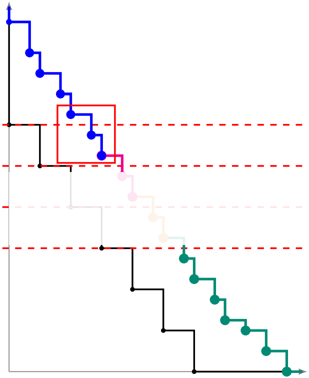
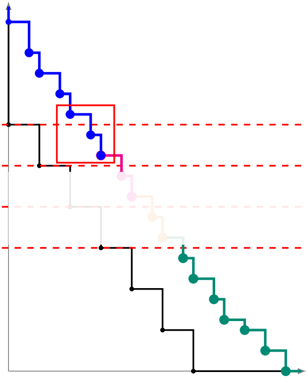
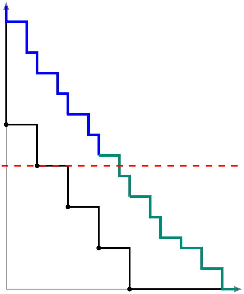


One summand $(x^u, y^v)^k J$

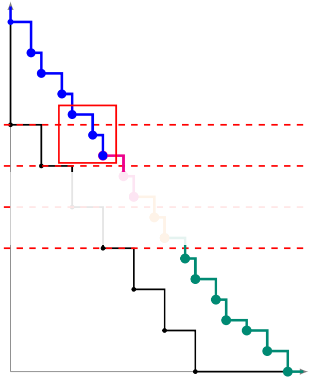


$k = r + 3$

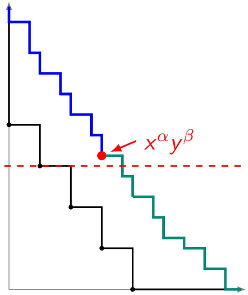


$k = r + 1$

One summand $(x^u, y^v)^k J$



$k = r + 3$



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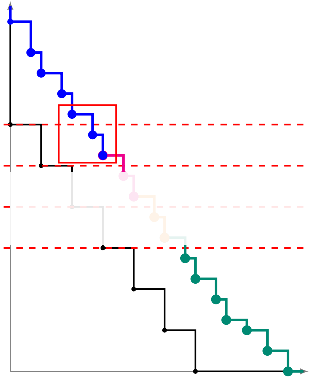

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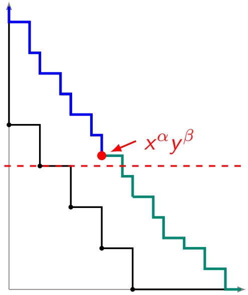
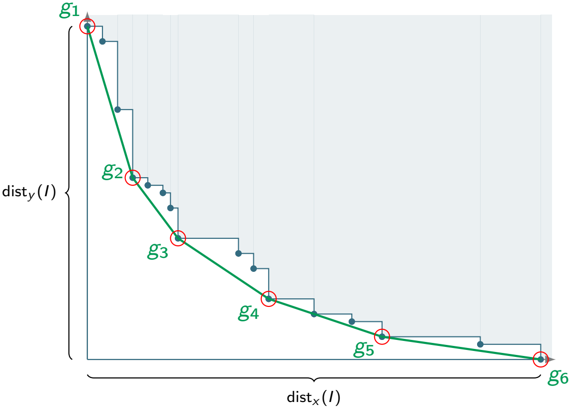


$k = r + 1$

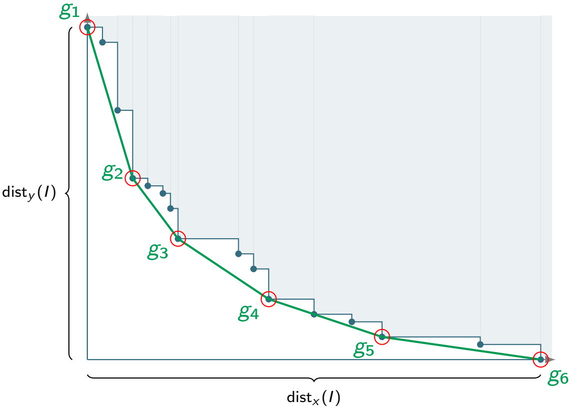
Theorem (Rath, R., 2025)

For all $r \geq \left\lceil \frac{\mathrm{dist}_y J}{v} \right\rceil$ and $\ell \in \mathbb{N}_0$: $(x^u, y^v)^{r+1+\ell} J = A \circledcirc_y H^{\circledcirc_y \ell} \circledcirc_y B$.



Theorem (Rath, R., 2025)

$$\forall \ell \geq 0\colon\ I^{D+\ell} = \sum_{i=1}^{k} (g_i, g_{i+1})^{\ell} I^D \quad \text{with } D = \mu(I)\,\mathrm{dist}_\bullet(I)^2$$



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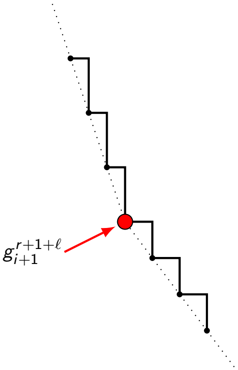
Two summands



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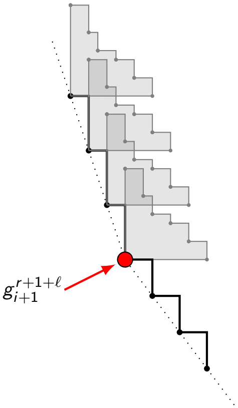

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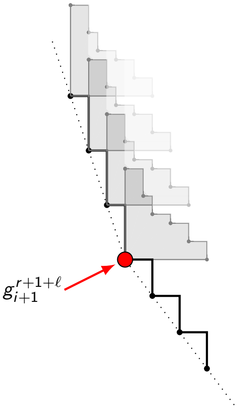

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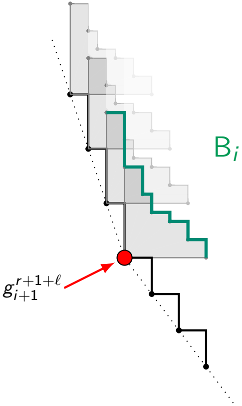

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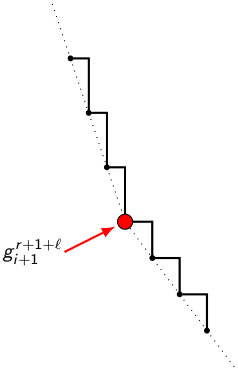

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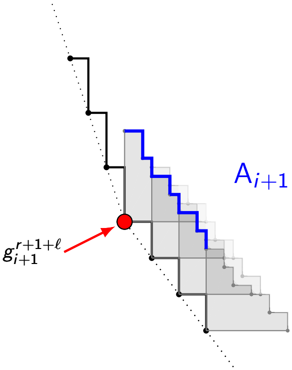

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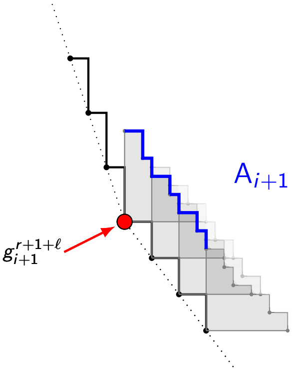

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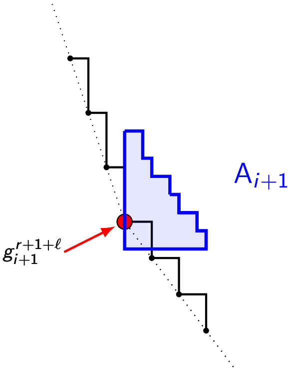
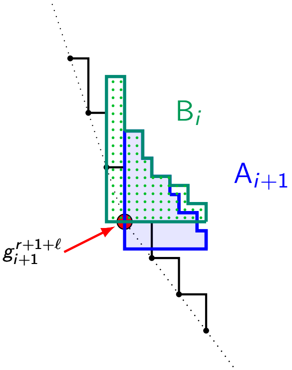


Two summands



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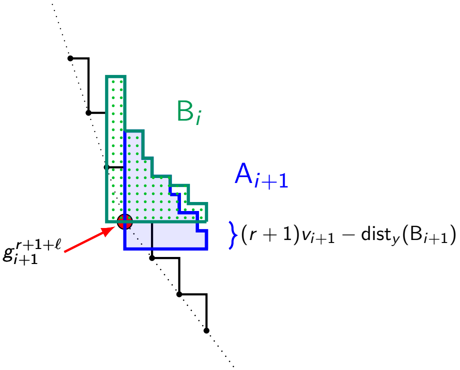

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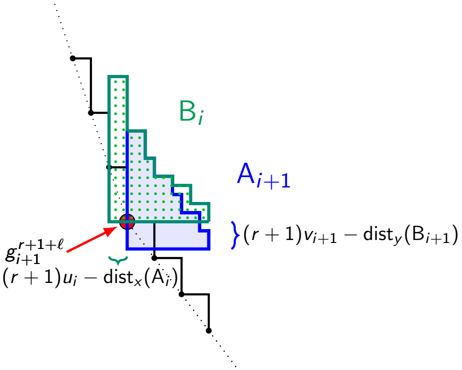

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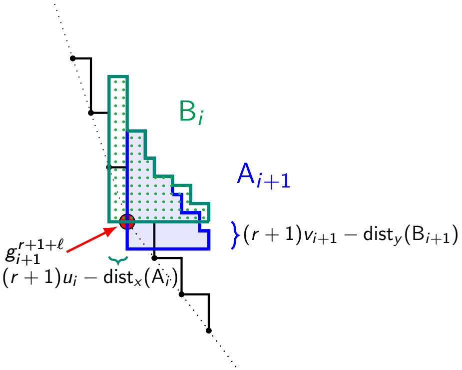
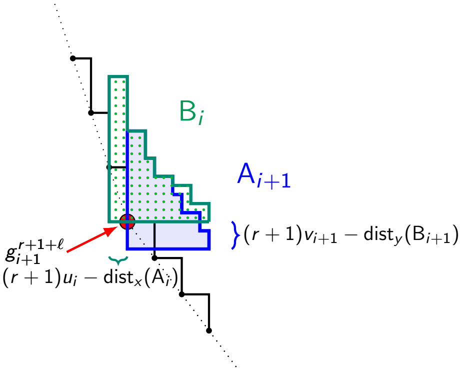


$$C_i := B_i \cdot y^{(r+1)v_{i+1} - \mathrm{dist}_y(B_{i+1})} + A_{i+1} \cdot x^{(r+1)u_i - \mathrm{dist}_x(A_i)}$$

Two summands



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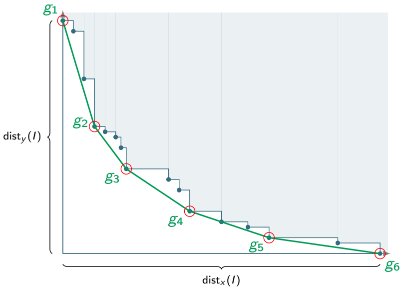
Theorem (Rath, R., 2025)

$$\sum_{i=1}^{k} (g_i, g_{i+1})^{r+1+\ell} J = C_0 \circledcirc_y \bigodot_{i=1}^{k} {}_y \left(H_i^{\circledcirc_y \ell} \circledcirc_y C_i \right),$$

Summing up

This research was funded in part by the Austrian Science Fund (FWF) [10.55776/DOC78].

Summing up



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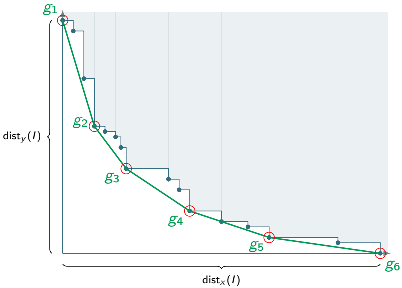
Summing up



Theorem (Rath, R., 2025)

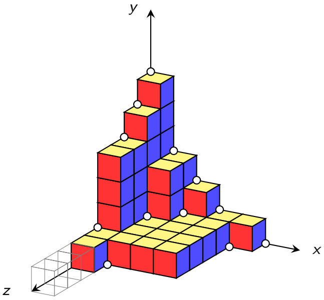
If $s \geq \mu(I)\,\mathrm{dist}_y(I)^2$ then

$$\forall \ell \geq 0\colon\ I^{s+\ell} = \gcd(I)^{s+\ell} \cdot \left(\mathrm{C}_0 \mathbin{\textcircled{\scriptsize y}} \mathop{\textcircled{y}}_{i=1}^{k} \left(\mathrm{H}_i^{\textcircled{\scriptsize y}\ell} \mathbin{\textcircled{\scriptsize y}} \mathrm{C}_i \right) \right)$$

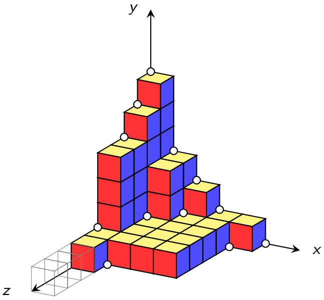
This research was funded in part by the Austrian Science Fund (FWF) [10.55776/DOC78].

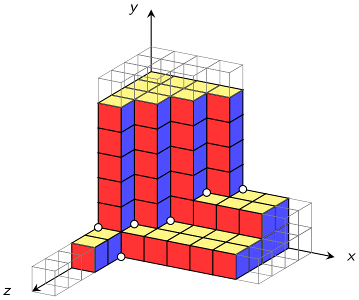
Motivation: Associated primes in 3D

Searching corners



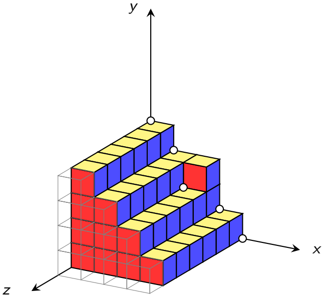
Searching corners





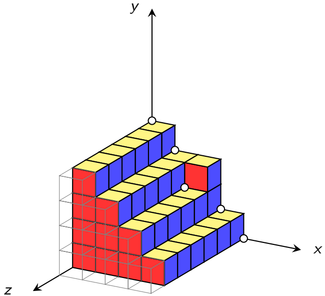
Corners & powers

Corners & powers



$I = (x^4, y^4, x^3y, xy^3, x^2y^2z)$

Corners & powers



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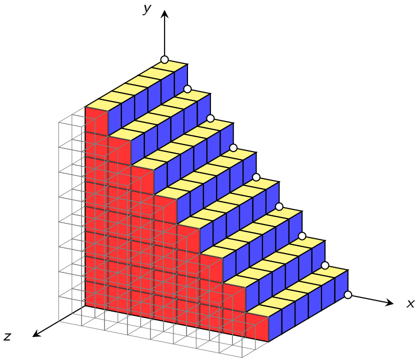

$I^2 = (x^4, y^4, x^3y, xy^3, x^2y^2z)^2$

Corners & powers II

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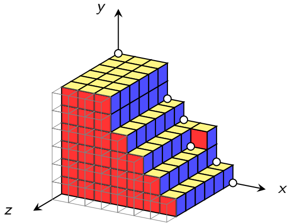
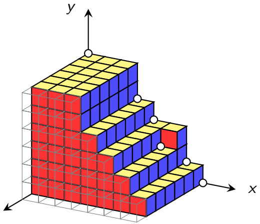


$I = (x^6, x^4y^3, y^7, x^3y^4, xy^7, x^2y^5z)$

Corners & powers II



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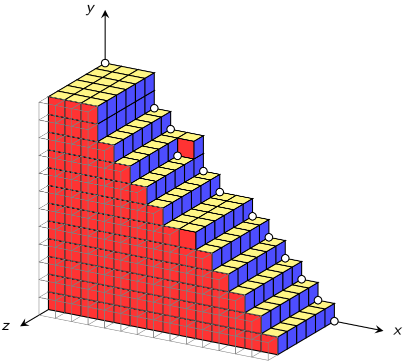

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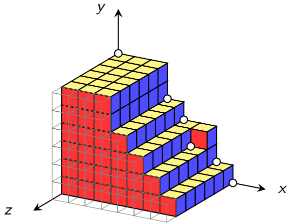

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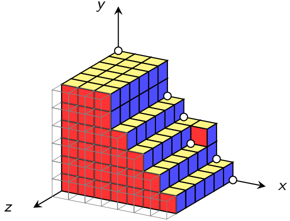

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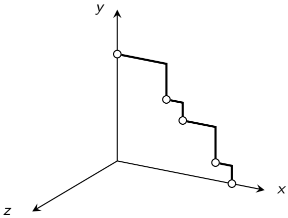

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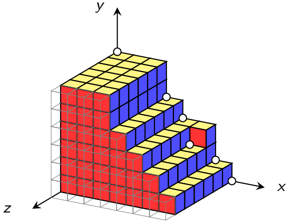

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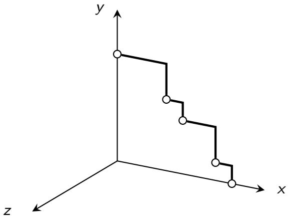
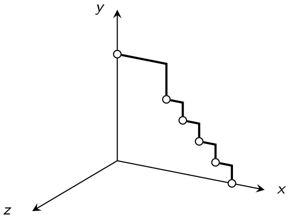
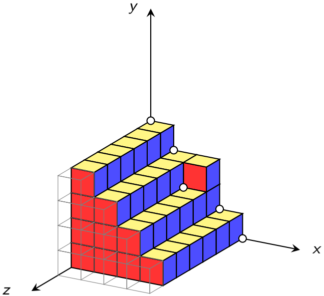




Corners & powers



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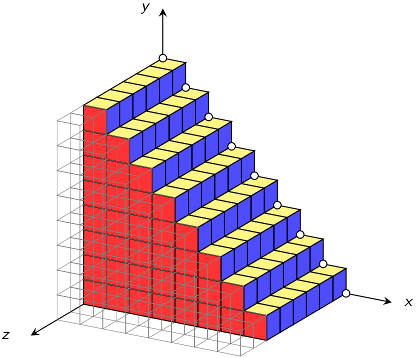

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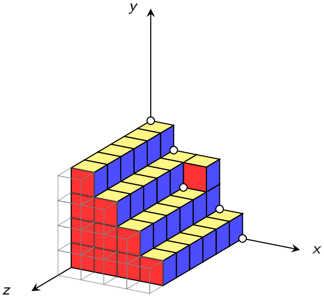
Corners & powers



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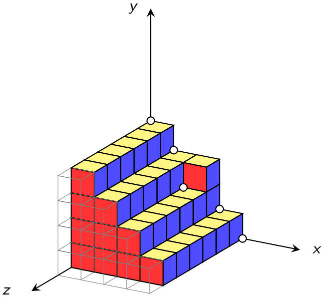

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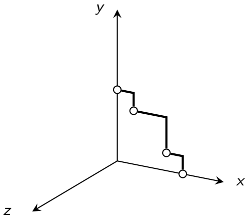

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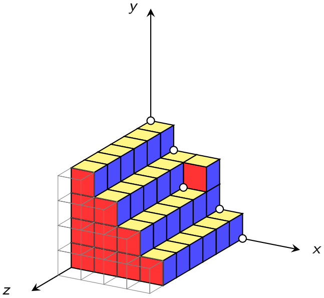

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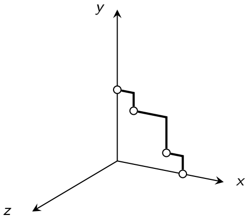

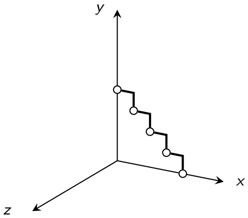

Theorem (Rath, R., 2025)

$I \subseteq k[x,y,z]$ monomial ideal with at most two associated primes of height 2, $d = \max_{\bullet \in \{x,y,z\}} \operatorname{dist}_\bullet I$.

Then for all $n, m \geq \mu(I)\left(d^2 - 1\right) + 2$,

$$\operatorname{Ass}(R/I^n) = \operatorname{Ass}(R/I^m)$$

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Conjecture.

~~Theorem~~ (Rath, R., 2025)

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Runtimes

		preprocessing		$s+10^2$	$s+10^3$	$s+10^4$	$s+10^5$	$s+10^6$
		I^D	I^s			$I^{s+\ell}$		
		$D = 13$	$s = 45$					
I_1	this method	0.005	0.01	0.04	0.35	4.30	51.35	584.89
	M2 with 3.11	0.0006	$*$	0.05	1.69	1503.34	–	–
	M2 (built-in)	$*$	$*$	0.08	22.25	34898.4	–	–
		$D = 40$	$s = 241$					
I_2	this method	0.12	0.30	0.13	0.69	7.44	87.12	980.50
	M2 with 3.11	0.02	$*$	0.36	8.53	3152.86	–	–
	M2 (built-in)	$*$	$*$	8.02	411.03	–	–	–
		$D = 76$	$s = 989$					
I_3	this method	0.73	6.29	0.52	1.49	12.12	139.68	1551.92
	M2 with 3.11	0.18	$*$	15.39	69.82	18724.5	–	–
	M2 (built-in)	$*$	$*$	607.32	5050.71	–	–	–
		$D = 238$	$s = 2064$					
I_4	this method	28.45	47.13	2.28	4.15	20.53	209.50	2305.38
	M2 with 3.11	84.81	$*$	71.77	176.13	13546.8	–	–
	M2 (built-in)	$*$	$*$	> 12h	–	–	–	–