Infinite direct-sum decompositions via a monoid-theoretical approach

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Motivation

Let R be a ring (unital, associative).

Then

$$\mathsf{V}(R) = \{\, [M] : M_R \text{ finitely generated projective} \,\}$$

is a monoid with $[M] + [N] = [M \oplus N]$.

- $\mathsf{V}(R)$ is **commutative**, **reduced** ($[M] + [N] = 0 \Rightarrow [M] = [N] = 0$), and has an **order-unit** (for every $[M]$ there exists $[N]$ and $k \geq 0$ such that $[M] + [N] = k[R]$).
- Finite direct sum-decompositions translate into indecomposables translate into factorizations into irreducibles (atoms) in $\mathsf{V}(R)$:

$$M \cong U_1 \oplus \cdots \oplus U_n \quad \Leftrightarrow \quad [M] = [U_1] + \cdots + [U_n].$$

Motivation

Theorem (Bergman, Bergman–Dicks '70s)

For every reduced commutative monoid H with order-unit, there exists a hereditary algebra R such that $V(R) \cong H$.

Example

Does there exist a ring R having a module M that has only two factorizations into irreducibles, one with 7 summands, one with 2025?

Yes, because in the numerical monoid $\langle 7, 2025 \rangle \subseteq \mathbb{N}_0$,

$$2025 \cdot 7 = \underbrace{2025 + \cdots + 2025}_{7\times} = \underbrace{7 + \cdots + 7}_{2500\times}.$$

Motivation

$V^*(R)$ is the monoid of **countably generated projective modules**

Recently studied by Álvarez, Herbera, Příhoda, and R. Wiegand.

C.g. projectives are closed under **countable** $\oplus$, but this structure is missing in the monoid $V^*(R)$.

Countable $\oplus$ induces an operation on $V^*(R)$. Can we reasonably describe it by axioms?

κ-monoids

Let κ be an infinite cardinal.

Definition

A κ-monoid is a set H together with an element $0 \in H$ and a map $\Sigma \colon H^\kappa \to H$ such that the following conditions are satisfied.

1. If $x = (x_i)_{i \in \kappa} \in H^\kappa$ with $x_i = 0$ for all $i \neq 0$, then $\sum_{i \in \kappa} x_i = x_0$.
2. If $(x_{i,j})_{i,j \in \kappa} \in H^{\kappa \times \kappa}$ and $\pi \colon \kappa \times \kappa \to \kappa$ is a bijection, then

$$\sum_{i \in \kappa} \sum_{j \in \kappa} x_{i,j} = \sum_{k \in \kappa} x_{\pi^{-1}(k)}.$$

- Every κ-monoid is a λ-monoid for $\lambda \leq \kappa$ (and a monoid).
- Every κ-monoid is a commutative monoid.
- Notation $\sum_{i \in \kappa} x_i \coloneqq \Sigma((x_i)_{i \in \kappa})$.

κ-monoids: Examples

1. $\mathbb{N}_0 \cup \{\infty\}$ with
$$\sum_{i\in\kappa} x_i = \begin{cases} x_1 + \cdots + x_{n_0} & \text{if } x_i \in \mathbb{N}_0 \text{ and } x_n = 0 \text{ for } n > n_0 \\ \infty & \text{otherwise.} \end{cases}$$
("trivial extension")
2. $\mathbb{R}_{\geq 0} \cup \{\infty\}$ with trivial extension (here $\sum_{n=1}^{\infty} \frac{1}{n^2} = \infty$).
3. $\mathbb{R}_{\geq 0} \cup \{\infty\}$ with operation induced from convergent series (here $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$).
4. Cardinals $\{\,\lambda : \lambda \leq \kappa\,\}$ with cardinal arithmetic.
5. $\mathsf{V}^{\kappa}(R)$, projective modules generated by $\leq \kappa$-many elements: $\mathsf{V}^{\aleph_0}(R) = V^*(R)$.
6. $\mathsf{V}(\mathcal{C})$ with $\mathcal{C}$ a set of isomorphism classes of R-modules closed under $\Subset$, $\cong$, and direct sums over index sets of cardinality $\leq \kappa$.

Basic properties

Lemma

Every κ-monoid is reduced.

Proof by Eilenberg swindle.

Suppose $a + b = 0$. Show: $a = b = 0$.

$$0 = \aleph_0(a + b) = \aleph_0 a + \aleph_0 b = a + \aleph_0 a + \aleph_0 b = a + \aleph_0(a + b) = a + 0 = a. \qquad \square$$

Braiding property

Assume $\mathsf{V}^{\aleph_0}(R)$ is generated by $\mathsf{V}(R)$, that is, every projective R-module is a direct sum of finitely generated modules. (E.g., R hereditary).

Suppose

$$X_1 \oplus X_2 \oplus X_3 \oplus \cdots \quad \cong \quad Y_1 \oplus Y_2 \oplus Y_3 \oplus \cdots \qquad\qquad (X_i,\ Y_j \text{ f.g.})$$

$\Rightarrow\ \exists n_1$ such that $U_0 \coloneqq X_1 \subseteq Y_1 \oplus \cdots \oplus Y_{n_1}$.
$\Rightarrow\ X_1 \in_{\oplus} Y_1 \oplus \cdots \oplus Y_{n_1} \quad \Rightarrow \quad Y_1 \oplus \cdots \oplus Y_{n_1} = X_1 \oplus V_1$.

Same game with V_1: $\quad V_1 \in_{\oplus} X_2 \oplus \cdots \oplus X_{m_1}, \quad V_1 \oplus U_1 = X_2 \oplus \cdots \oplus X_{m_1}$.

Next: $U_1 \oplus V_2 = Y_{n_1+1} \oplus \cdots \oplus Y_{n_2}$, and so on.

Braiding property

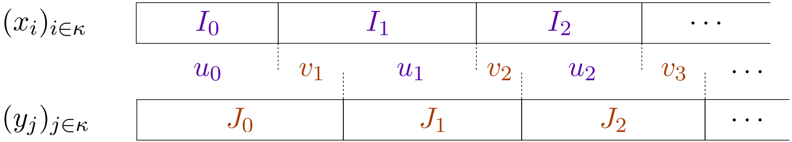
Definition

Let X be a monoid.

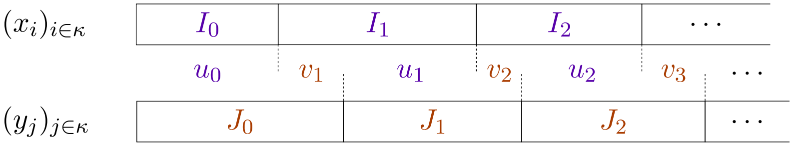
1. Families $(x_i)_{i\in\mathbb{N}_0}$ and $(y_j)_{j\in\mathbb{N}_0}$ in X are **($\aleph_0^-$)-braided** if there exist indexed partitions $(I_\mu)_{\mu\in\mathbb{N}_0}$ and $(J_\mu)_{\mu\in\mathbb{N}_0}$ of $\mathbb{N}_0$ with $|I_\mu|$, $|J_\mu|$ finite, and families $(u_\mu)_{\mu\in\mathbb{N}_0}$ and $(v_\mu)_{\mu\in\mathbb{N}_0}$ in X such that $v_0 = 0$, and

$$\sum_{i\in I_\mu} x_i = v_\mu + u_\mu, \qquad \text{and} \qquad \sum_{j\in J_\mu} y_j = v_{\mu+1} + u_\mu \qquad \text{for all } \mu \in \mathbb{N}_0.$$

2. Let H be an $\aleph_0$-monoid an X a submonoid. Then H is **($\aleph_0^-$)-braided over** X if $H = \langle X\rangle_{\aleph_0}$, and all families $(x_i)_{i\in\mathbb{N}_0}$, $(y_j)_{j\in\mathbb{N}_0}$ in X with $\sum_{i\in\mathbb{N}_0} x_i = \sum_{j\in\mathbb{N}_0} y_j$ are $\aleph_0^-$-braided.



Braiding property



Theorem (Nazemian-S. '24)

1. If every projective module is a direct sum of f.g. projective ones, then $\mathsf{V}^{\aleph_0}(R)$ is $\aleph_0^-$-braided over $V(R)$.
2. In general, $\langle V(R)\rangle_{\aleph_0} \subseteq \mathsf{V}^{\aleph_0}(R)$ is $\aleph_0^-$-braided over $V(R)$.
3. $\mathsf{V}^{\kappa}(R)$ is λ^--braided over $\mathsf{V}^{\lambda}(R)$ for $\kappa > \lambda > \aleph_0$.

Theorem (Nazemian-S. '24)

Let $\mathcal{C}$ be a class of modules, closed under **countable** $\oplus$, $\Subset$, $\cong$ and $\mathcal{C}_{\mathsf{fg}}$ the class of finitely generated submodules.

If $\mathsf{V}^{\aleph_0}(\mathcal{C}) = \langle \mathsf{V}(\mathcal{C}_{\mathsf{fg}})\rangle_{\aleph_0}$, then $\mathsf{V}^{\aleph_0}(\mathcal{C})$ is $\aleph_0^-$-braided over $\mathsf{V}(\mathcal{C}_{\mathsf{fg}})$.

Universal $\aleph_0$-extensions

Definition (Universal Property)

Let H be a monoid. An $\aleph_0$-monoid $\widehat{H}$ is a **universal $\aleph_0$-extension** of H if there is a monoid homomorphism $\iota\colon H \to \widehat{H}$ satisfying: for every monoid homomorphism $\varphi\colon H \to K$ to a $\aleph_0$-monoid K, there exists a unique $\aleph_0$-homomorphism $\widehat{\varphi}\colon \widehat{H} \to K$ such that $\varphi = \widehat{\varphi} \circ \iota$.

$$\begin{tikzcd} H \arrow[r, "\iota"] \arrow[dr, "\varphi"] & \widehat{H} \arrow[d, "\widehat{\varphi}"] \\ & K. \end{tikzcd}$$

- Unique up to unique isomorphism.
- Existence is non-trivial but can be proven.

Theorem (Nazemian-S. '24)

Let H be a $\aleph_0$-monoid and X be a submonoid. TFAE

1. H is $\aleph_0^-$-braided over X.
2. H is the universal $\aleph_0$-extension of X.

Similar concepts for κ-monoids as extensions of λ^--monoids with $\lambda \leq \kappa$ a regular cardinal.

Example

- If $X = \mathbb{N}_0$ then $H = \mathbb{N}_0 \cup \{\infty\}$ is $\aleph_0^-$-braided over X.
- If $X = \mathbb{R}_{\geq 0}$, then none of the two $\aleph_0$-monoid structures we saw on $\mathbb{R}_{\geq 0} \cup \{\infty\}$ is $\aleph_0^-$-braided over X. The following is:

$$H \cong \mathbb{R}_{\geq 0} \cup \widetilde{\mathbb{R}}_{>0} \cup \{\infty\}.$$

Projective modules

Every projective module is a direct sum of countably generated projective modules (Kaplansky).

Corollary

Let R be a ring.

1. If every projective module is a direct sum of finitely generated projective modules (e.g., if R is hereditary), then $\mathsf{V}^{\aleph_0}(R)$ is the universal $\aleph_0$-extension of $\mathsf{V}(R)$.
2. In any case, $\mathsf{V}^{\kappa}(R)$ for $\kappa > \aleph_0$ the universal κ-extension of $\mathsf{V}^{\aleph_0}(R)$.

$\mathsf{V}^{\kappa}(R)$ is fully determined by $\mathsf{V}^{\aleph_0}(R)$!

Realizability

Corollary

1. The κ-monoids realizable as $\mathsf{V}^{\kappa}(R)$ with R a hereditary ring are precisely the universal κ-extensions of reduced commutative monoids with order-unit.
2. The κ-monoids realizable as $\mathsf{V}^{\kappa}(R)$ for arbitrary rings R are precisely the universal κ-extensions of the **realizable** $\aleph_0$-monoids.

Open Question: Which $\aleph_0$-monoids appear as $V^{\aleph_0}(R)$ for some ring?

$\mathsf{V}^*(R)$ and $\mathsf{V}^\kappa(R)$

Herbera and Příhoda fully characterized $\mathsf{V}^*(R)$ for semilocal noetherian commutative rings as submonoids of $(\mathbb{N}_0 \cup \{\infty\})^k$ defined by **linear equations** and **linear inequalities**.

$\mathsf{V}(R)$ does not fully determine $\mathsf{V}^{\aleph_0}(R)$, e.g., $x = y$ and $2x = x + y$ both define the submonoid

$$\{(a,a) : a \in \mathbb{N}_0\} \subseteq \mathbb{N}_0^2,$$

but in $(\mathbb{N}_0 \cup \{\infty\})^2$, the first equation defines

$$\{(a,a) : a \in \mathbb{N}_0 \cup \{\infty\}\},$$

and the second

$$\{(a,a) : a \in \mathbb{N}_0\} \cup \{(\infty,a) : a \in \mathbb{N}_0\}.$$

Corollary

$\mathsf{V}^{\aleph_0}(R)$ fully determines $\mathsf{V}^\kappa(R)$ — we can use same equalities/inequalities to define it!

Conclusion

1. With κ-monoids we can model infinite direct sums on the monoid side.

2. Direct-sums give rise to the non-trivial **braiding property** that corresponds to **universal property** for κ-extensions.

3. In several interesting cases therefore $\mathsf{V}(R)$ (or at least $\mathsf{V}^{\aleph_0}(R)$) fully determines $\mathsf{V}^{\kappa}(R)$.