Versatility of locally reduced modules

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Abstract

Let R be a unital commutative ring and $I \subseteq R$ a fixed ideal. Just like local cohomology R-modules with respect to I, I-reduced R-modules form a notion which allows the study of properties of R-modules in relation to the ideal I. We demonstrate the versatility of I-reduced R-modules. For instance, they have applications in category theory, in torsion theory and in homological algebra - more specifically, in the study of local cohomology modules. We also propose some research problems.

Outline

- The introduction
- Applications in category theory
- Applications in radical theory and torsion theory
- Relationship with local cohomology
- Research questions in commutative algebra, algebraic geometry and in non-commutative algebra
- References

A. The introduction

Reduced modules

- A ring R is *reduced* if for all ideals I of R, $I^2 = 0$ implies that $I = 0$.
- An R-module M is reduced if for all ideals I of R, and all $m \in M$, $I^2 m = 0$ implies that $Im = 0$.

I-reduced modules

- For a fixed ideal I of R, we say that an R-module M is I-reduced if for all $m \in M$, $I^2 m = 0$ implies that $Im = 0$.
- If M is I-reduced for some ideal I of R, we say that M is *locally reduced*.
- All reduced modules are locally reduced but not conversely.

Introduction continued

- So, an R-module is reduced if it is I-reduced for all ideals I of R.
- Reduced modules were introduced by Lee and Zhou in [5].
- Locally reduced modules have been studied in [4, 7, 8, 9] among others.

Examples

For any R-module M and any ideal I of R, the following R-modules are I-reduced (they need not be reduced)

- M/IM
- $(0:_M I) := \{m \in M \ : \ Im = 0\}$.

This demonstrates that I-reduced modules form a large class of modules.

- A ring R is reduced (resp. I-reduced) iff it is reduced (resp. I-reduced) as an R-module.
- Any projective module over an I-reduced ring is I-reduced.
- A torsionfree module is reduced.

B. Application in category theory

- The entry point into category theory is via the I-torsion functor.
- The I-torsion functor Γ_I associates to every R-module M, a submodule

$$\Gamma_I(M) := \{m \in M \ : \ I^k m = 0 \text{ for some } k \in \mathbb{Z}^+\}.$$

$$\Gamma_I(M) = \bigcup_{k \in \mathbb{Z}^+} (0 :_M I^k) \cong \varinjlim_k \operatorname{Hom}_R(R/I^k, M)$$

- This functor can be used to characterise I-reduced R-modules.

Equivalent statements for I-reduced modules

Proposition 2.1

For any R-module M and an ideal I of R, the following statements are equivalent:

1. M *is* I*-reduced,*
2. $(0:_M I) = (0:_M I^2)$,
3. $Hom_R(R/I, M) \cong Hom_R(R/I^2, M)$,
4. $\Gamma_I(M) \cong Hom_R(R/I, M)$,
5. $I\Gamma_I(M) = 0$.

I-coreduced modules and coreduced modules

I-coreduced modules

An R-module M is I-coreduced if $I^2M = IM$.

Coreduced modules

An R-module M is coreduced if it is I-coreduced for all ideals I of R.

1. Divisible modules are coreduced.
2. A functor dual Γ_I is the I-adic completion functor. It is given by

$$\Lambda_I(M) := \varprojlim_k (R/I^k \otimes_R M).$$

3. It is used to characterise I-coreduced modules, a notion dual to I-reduced modules.

Equivalent statements for I-coreduced modules

Proposition 2.2

For any R-module M and an ideal I of R, the following statements are equivalent:

1. M *is* I*-coreduced,*
2. $R/I \otimes M \cong R/I^2 \otimes M$,
3. $\Lambda_I(M) \cong R/I \otimes M$,
4. $I\Lambda_I(M) = 0$.

Greenlees-May Duality

- The functors Γ_I and Λ_I are in general not adjoint.
- In the setting of derived categories, when I is a weakly proregular ideal, we have the Greenlees-May Duality (GM Duality for short).

Theorem 2.1 (Tarrio, Lopez and Lipman)

Let I be a weakly proregular ideal of a ring R and $M, N \in \mathbf{D}(R)$. Then there is a natural isomorphism in $\mathbf{D}(R)$ given by

$$\mathbf{R}Hom_R(\mathbf{R}\Gamma_I(M), N) \cong \mathbf{R}Hom_R(M, \mathbf{L}\Lambda_I(N)).$$

- Local cohomology is derived left adjoint to local homology.

GM Duality in R-Mod

Theorem 2.2 (D. Ssevviiri, 2023)

For any ideal I of a ring R,

1. *The functor* $\Gamma_I : (R\text{-}Mod)_{I\text{-}red} \to (R\text{-}Mod)_{I\text{-}cor}$ *is idempotent and for any* $M \in (R\text{-}Mod)_{I\text{-}red}$, $\Gamma_I(M) \cong Hom_R(R/I, M)$.
2. *The functor* $\Lambda_I : (R\text{-}Mod)_{I\text{-}cor} \to (R\text{-}Mod)_{I\text{-}red}$ *is idempotent and for any* $M \in (R\text{-}Mod)_{I\text{-}cor}$, $\Lambda_I(M) \cong R/I \otimes M$.
3. *For any* $N \in (R\text{-}Mod)_{I\text{-}red}$ *and* $M \in (R\text{-}Mod)_{I\text{-}cor}$,
$$Hom_R(\Lambda_I(M), N) \cong Hom_R(M, \Gamma_I(N)).$$

In this setting, Λ_I is left-adjoint to Γ_I.

MGM Equivalence

- An R-module M is I-torsion if $\Gamma_I(M) = M$.
- An R-module M is I-complete if $\Lambda_I(M) \cong M$.
- A collection of all I-torsion R-modules is an abelian category.
- A collection of all I-complete R-modules need **NOT BE** an abelian category.
- So, these two subcategories are not equivalent in general.
- In the derived category setting with I weakly proregular, we get an equivalence.
- This is the so called Matlis-Greenlees-May Equivalence (MGM Equivalence for short).

MGM Equivalence continued

Theorem 2.3 (Porta, Shaul, Yekutieli, 2014)

Let R be a ring, and let I be a weakly proregular ideal in it.

1. *For any* $M \in \mathbf{D}(R)$,

   $\mathbf{R}\Gamma_I(M) \in \mathbf{D}(R)_{I\text{-}tor}$ *and* $\mathbf{L}\Lambda_I(M) \in \mathbf{D}(R)_{I\text{-}com}$.

2. *The functor*

$$\mathbf{R}\Gamma_I \; : \; \mathbf{D}(R)_{I\text{-}com} \to \mathbf{D}(R)_{I\text{-}tor}$$

   is an equivalence, with quasi-inverse $\mathbf{L}\Lambda_I$.

The MGM Equivalence in R-Mod

Theorem 2.4 (D. Ssevviiri, 2023)

Let I be any ideal of a ring R,

1. *For any $M \in (R\text{-}Mod)_{I\text{-}red}$,*

$$\Gamma_I(M) \in (R\text{-}Mod)_{I\text{-}com} \cap (R\text{-}Mod)_{I\text{-}cor} =: \mathfrak{E}.$$

2. *For any $M \in (R\text{-}Mod)_{I\text{-}cor}$,*

$$\Lambda_I(M) \in (R\text{-}Mod)_{I\text{-}tor} \cap (R\text{-}Mod)_{I\text{-}red} =: \mathfrak{D}.$$

3. *The functor $\Gamma_I : (R\text{-}Mod)_{I\text{-}red} \to (R\text{-}Mod)_{I\text{-}cor}$ restricted to $\mathfrak{D}$ is an equivalence between $\mathfrak{D}$ and $\mathfrak{E}$ with quasi-inverse Λ_I.*
4. *The equivalence is actually an equality.*

Computation of natural transformations

Since Γ_I is representable on the subcategory of I-reduced R-modules, we can invoke the Yoneda Lemma to compute natural transformations between Γ_I and other functors. For instance, we have

Proposition 2.3

For any ideal I of a ring R, and functors

$$\Gamma_I : (R\text{-}Mod)_{I\text{-}red} \to (R\text{-}Mod)_{I\text{-}cor} \quad \textit{and} \quad I \otimes - : (R\text{-}Mod)_{I\text{-}red} \to \mathbf{Set},$$

we have

$$Nat(\Gamma_I(-), \Gamma_I(-)) \cong R/I \quad \textit{and} \quad Nat(\Gamma_I(-), I \otimes -) \cong 0.$$

C. Radicals and torsion theory

The objective of this section

We give the necessary and sufficient conditions in terms of I-reduced and I-coreduced R-modules for the functor $\text{Hom}_R(R/I, -)$ on the abelian full subcategory of the category of R-modules to be a radical. These conditions further provide a setting for the generalisation of Jans' correspondence, and lead to a new radical class of rings.

Definition of a (pre)radical

A functor $\gamma : R\text{-Mod} \to R\text{-Mod}$ which associates to every R-module M, a submodule $\gamma(M)$ of M is a:

(i) *preradical* if for every R-homomorphism $f : M \to N$, $f(\gamma(M)) \subseteq \gamma(N)$;

(ii) *radical* if it is a preradical and for all $M \in R\text{-Mod}$, $\gamma(M/\gamma(M)) = 0$.

Examples of radicals

1. For any R-module M, the intersection of all maximal submodules of M is radical called the *Jacobson radical* of M.
2. Let I be an ideal of R. The functor $\delta_I : R\text{-Mod} \to R\text{-Mod}$ which associates to every R-module M, a submodule IM, is a radical.
3. For any finitely generated ideal I of a ring R, the I-torsion functor Γ_I is a left exact idempotent radical on R-Mod.
4. Let S be a multiplicatively closed subset of an integral domain R, the submodule

$$t(M) := \{m \in M \ : \ sm = 0 \text{ for some } s \in S\}$$

of M, defines a left exact idempotent radical of R-Mod.
5. For any R-module M, the Bass torsion $B(M) := \text{Ker}(M \to M^{**})$, where $M^* := \text{Hom}_R(M, R)$, is a radical.

Properties of the functor $\text{Hom}_R(R/I, -)$

Proposition 3.1

For any ideal I of a ring R, the functor $Hom_R(R/I, -)$ on the category R-Mod is a preradical.

Lemma 3.1

Let I be an ideal of a ring R. If the functor Γ_I is a radical, then it is the smallest radical containing the preradical $Hom_R(R/I, -)$.

Proposition 3.2

For any ideal I of a ring R, let $\mathcal{A}_I$ and $\mathcal{B}_I$ be abelian full subcategories of R-Mod.

1. *The functor $Hom_R(R/I, -)$ on $\mathcal{A}_I$ is a radical if and only if $\mathcal{A}_I$ consists of I-reduced R-modules.*
2. *The radical δ_I which associates to every R-module M in $\mathcal{B}_I$, an R-submodule $\delta_I(M) := IM$ is idempotent if and only if $\mathcal{B}_I$ consists of I-coreduced R-modules.*

Torsion theory

A *torsion theory* τ for an abelian category $\mathcal{C}$ is a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects of $\mathcal{C}$ such that

1. $\text{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$;
2. if $\text{Hom}(A, F) = 0$ for all $F \in \mathcal{F}$, then $A \in \mathcal{T}$;
3. if $\text{Hom}(T, B) = 0$ for all $T \in \mathcal{T}$, then $B \in \mathcal{F}$.

- $\mathcal{T}$ is called the *torsion class* of τ
- $\mathcal{F}$ is called the *torsionfree class* of τ
- A class $\mathcal{H}$ of an abelian category $\mathcal{C}$ is a *torsion-torsionfree* class (TTF class) if it is both a torsion class and a torsionfree class.
- A torsion class is *hereditary* if it is closed under taking submodules.

Jans' Correspondence

Theorem 3.1 (Jans, 1965)

There is a one-to-one correspondence between an idempotent ideal I of R and the TTF class $\{M \in R\text{-}Mod\ : IM = 0\}$.

The Generalised Jans' Correspondence

Theorem 3.2 (D. Ssevviiri, 2025)

For any ideal I of a ring R and an abelian full subcategory $\mathcal{A}_I$ (resp. $\mathcal{B}_I$) of R-Mod consisting of I-reduced (resp. I-coreduced) R-modules such that $\Gamma_I : \mathcal{A}_I \to \mathcal{B}_I$ and $\Lambda_I : \mathcal{B}_I \to \mathcal{A}_I$ form an adjoint pair, the following hold.

1. *The torsion theory associated to the radical Γ_I is given by $T_I := \{M \in \mathcal{A}_I \ : \ \Gamma_I(M) = M\}$ and $F_I := \{M \in \mathcal{A}_I \ : \ \Gamma_I(M) = 0\}$.*
2. *T_I is a TTF class.*
3. *$\mathfrak{T}_I$ for which $(\mathfrak{T}_I, T_I)$ is a torsion theory is given by $\mathfrak{T}_I := \{M \in \mathcal{B}_I \ : \ IM = M\}$ with the associated idempotent radical δ_I on $\mathcal{B}_I$ given by $\delta_I(M) := IM$.*
4. *There is a one-to-one correspondence between the abelian full subcategory $\mathcal{A}_I$ and the TTF class T_I.*

Necessary and sufficient conditions for radicality

Theorem 3.3 (D. Ssevviiri, 2025)

Let I be an ideal of a ring R and let $\mathcal{A}_I$ and $\mathcal{B}_I$ be abelian full subcategories of R-Mod such that the functors $\Gamma_I : \mathcal{A}_I \to \mathcal{B}_I$ and $\Lambda_I : \mathcal{B}_I \to \mathcal{A}_I$ form an adjoint pair. The following statements are equivalent:

1. *the functor $\mathrm{Hom}_R(R/I, -) : \mathcal{A}_I \to \mathcal{B}_I$ is a radical;*
2. *$\mathcal{A}_I$ consists of I-reduced R-modules;*
3. *$T_I := \{M \in \mathcal{A}_I \ : \ IM = 0\}$ is a TTF;*
4. *$\mathcal{B}_I$ consists of I-coreduced R-modules;*
5. *the radical δ_I which associates to every R-module M in $\mathcal{B}_I$, an R-submodule $\delta_I(M) := IM$ is idempotent.*

Corollary 3.1

Let I be an ideal of a ring R. The following statements are equivalent:

1. *The functor $Hom_R(R/I, -)$ is a radical on the category R-Mod.*
2. *Every R-module is I-reduced.*
3. *I is an idempotent ideal.*
4. $T_I := \{M \in R\text{-Mod} : IM = 0\}$ *is a TTF.*
5. *The functor $\delta_I(M) := IM$ is an idempotent radical on the category R-Mod.*
6. *Every R-module is I-coreduced.*

Examples

- If I is an idempotent ideal of R, then $\mathcal{A}_I = \mathcal{B}_I = R$-Mod and Theorem 3.2 retrieves Jans' correspondence.
- If R is a Noetherian ring and R-mod is the full subcategory of R-Mod consisting of all finitely generated R-modules, then every module $M \in R$-mod is I^k-reduced for some positive integer k. If $t = \underset{M \in R\text{-Mod}}{\text{Maximum}}\{k(M)\}$ exists, then $\mathcal{C}_{I^t} = R$-mod.
- If I is any ideal of R, then the collection of all semisimple R-modules forms an abelian category whose modules are both I-reduced and I-coreduced.

The radical class of rings induced

S is a ring which is not necessarily commutative or unital.

A class of rings Ψ is called a *radical class* if

1. Ψ is homomorphically closed, i.e., if $S \in \Psi$ and $f : S \to T$ is a ring homomorphism, then $f(S) \in \Psi$;
2. for every ring $S \in \Psi$, the sum $\Psi(S) := \sum\{J \lhd S \ : \ J \in \Psi\}$ is in Ψ;
3. $\Psi(S/\Psi(S)) = 0$ for all rings $S \in \Psi$.

Theorem 3.4 (D. Ssevviiri, 2025)

Let I be an idempotent ideal of a ring R. The class of rings

$$\Psi_I := \{S \ : \ S \text{ is a ring and an } R\text{-module such that } IS = 0\}$$

is a radical class.

D. Relationship with local cohomology

In this section, all rings will be Noetherian. For any R-module M and an ideal I of R,

- the ith local cohomology module with respect to I is the module

$$H_I^i(M) := H^i(\Gamma_I(E_M^*)),$$

  where E_M^* is the injective resolution of M.
- $H_I^i(M) \cong \varinjlim_k \operatorname{Ext}_R^i(R/I^k, M)$.
- The computation can also be via the Čech complex or the Koszul complex.
- Local cohomology has several applications in both commutative algebra and algebraic geometry.

Local cohomology continued

Local cohomology is used to characterise several notions, among them are:

- depth of a module,
- dimension of a module,
- Cohen-Macaulay modules,
- Gorenstein rings.

Depth and dimension

- The *depth of I on M*, denoted $\operatorname{depth}_R(I, M)$, is the maximal length of an M-regular sequence in I.
- When $(R, \mathfrak{m}, k)$ is a local ring, we simply write $\operatorname{depth}_R M$ for $\operatorname{depth}_R(\mathfrak{m}, M)$.
- The *dimension* of a finitely generated R-module M, denoted by $\dim_R(M)$, is the Krull dimension of the ring $R/(0 :_R M)$, where $(0 :_R M)$ is the annihilator of the R-module M.

Characterisation of depth and dimension

Theorem 4.1

Let $I \subseteq R$ be an ideal and M a finitely generated R-module such that $IM \neq M$, then

$$depth_R(I, M) = inf\ \{n \in \mathbb{N} \mid H_I^n(M) \neq 0\}.$$

Theorem 4.2

Let $(R, \mathfrak{m}, k)$ be a local ring and M be a finitely generated R-module.

$$dim_R(M) = sup\ \{n \in \mathbb{N} \mid H_{\mathfrak{m}}^n(M) \neq 0\}.$$

Characterisation of CM modules

Definition 4.1

Let $(R, \mathfrak{m}, k)$ be a local ring. A finitely generated R-module M is *Cohen-Macaulay* if $\operatorname{depth}_R(M) = \dim_R(M)$. The ring R is *Cohen-Macaulay* if it is Cohen-Macaulay as an R-module.

Theorem 4.3

Let $(R, \mathfrak{m}, k)$ be a local ring. A finitely generated R-module M is Cohen-Macaulay if and only if for all $i \neq dim_R(M)$,

$$H^i_{\mathfrak{m}}(M) = 0.$$

Characterisation of Gorenstein rings

Definition 4.2

A *Gorenstein local ring* is a commutative Noetherian local ring R with finite injective dimension as an R-module.

Theorem 4.4

Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. A ring R is Gorenstein if and only if

$$H^i_{\mathfrak{m}}(R) = \begin{cases} 0 & \text{for } i \neq d \\ E_R(k) & \text{for } i = d. \end{cases}$$

Local Duality

Theorem 4.5 (Local Duality 1)

Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring of dimension d and M be a finitely generated R-module. For $0 \leq i \leq d$, $H^i_{\mathfrak{m}}(M) \cong Ext^{d-i}_R(M, R)^v$, where $(-)^v = Hom_R(-, E_R(k))$.

Theorem 4.6 (Local Duality 2)

Let $(R, \mathfrak{m}, k)$ be a d-dimensional Cohen-Macaulay local ring with a canonical module ω. If M is a finitely generated R-module, then for $0 \leq i \leq d$, $H^i_{\mathfrak{m}}(M) \cong Ext^{d-i}_R(M, \omega)^v$, where $(-)^v = Hom_R(-, E_R(k))$.

Some notes about local duality

Local duality

- is the local avatar of Serre duality;
- allows for transfer of questions on local cohomology to questions about Ext modules.

E. Research questions

- If I is an idempotent ideal of R, then every R-module is I-reduced and

$$H_I^i(M) \cong \mathrm{Ext}_R^i(R/I, M).$$

- This simplifies the computation of local cohomology.

The first question - Commutative Algebra

Question 5.1

Does there exist a "nice" abelian subcategory $\mathcal{A}_I$ of R-Mod which has enough injectives and consists of I-reduced R-modules?

Importance of the first question

An affirmative answer is important because

1. It would follow that for all $M \in \mathcal{A}_I$, $H^i_I(M) \cong \text{Ext}^i_R(R/I, M)$ simplifying the computation of local cohomology in this setting.
2. Just like local duality, it would allow transfer of questions from the local cohomology module to just the Ext module.
3. The isomorphism $H^i_I(M) \cong \text{Ext}^i_R(R/I, M)$ would lead to an affirmative answer to the 6 finiteness questions about local cohomology modules posed by Huneke in [3].
4. It would also answer some open questions posed by D. Eisenbud, M. Mustata and M. Stillman in [2]. See Questions 6.1 and 6.2 in [2].

Questions posed by Huneke

Let M be a finitely generated R-module.

1. When are the R-modules $H^i_I(M)$
   - **(i)** Artinian?
   - **(ii)** finitely generated?
   - **(iii)** I-cofinite?
2. When are the R-modules $\mathrm{Soc}(H^i_I(M))$ finitely generated?
3. Is the set $\mathrm{Ass}_R(H^i_I(M))$ finite?
4. When are the Bass numbers of $H^i_I(M)$ finite?

The second question - Algebraic Geometry

- The functor Γ_I first appeared in algebraic geometry in the setting of sheaves where it is called the section functor.
- Indeed, there is a version of GM Duality and MGM Equivalence for schemes, see [1].

Question 5.2

Can we get a geometric interpretation (and geometric applications) of the notion of locally reduced modules?

The third question- Noncommutative Algebra

- Let R be a noncommutative ring. If I is a right ideal of R, then $I\Gamma_I(R)$ is a nil right ideal of R.
- This is a gadget which associates to every right ideal of R, a nil right ideal.
- Köthe conjecture states that the sum of two nil right ideals of R is nil.

Question 5.3

Can the aforementioned gadget be utilised to construct counter examples to the Köthe conjecture?

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The End

Thank you!