Rings and Modules that Satisfies the Radical Formula

Steven

School of Mathematical and Physical Sciences
University of Sheffield

Conference on Rings and Polynomials
TU Graz, Graz, Austria
15 July 2 025

Outline

Motivation

In this talk, R is a commutative ring and M is a unital R-module.

Motivation

In this talk, R is a commutative ring and M is a unital R-module.

Definition 1

Let R be a ring. An ideal P of R is **prime** if $rs \in P$ for $r, s \in R$ implies that $r \in P$ or $s \in P$.

Motivation

In this talk, R is a commutative ring and M is a unital R-module.

Definition 1

Let R be a ring. An ideal P of R is **prime** if $rs \in P$ for $r, s \in R$ implies that $r \in P$ or $s \in P$.

Definition 2

The **nilradical** of R is the set $N(R) := \{r \in R : r^n = 0 \text{ for some } n \in \mathbb{N}\}$.

Motivation

In this talk, R is a commutative ring and M is a unital R-module.

Definition 1

Let R be a ring. An ideal P of R is **prime** if $rs \in P$ for $r, s \in R$ implies that $r \in P$ or $s \in P$.

Definition 2

The **nilradical** of R is the set $N(R) := \{r \in R : r^n = 0 \text{ for some } n \in \mathbb{N}\}$.

Theorem 3

The nilradical of R is equal to the intersection of all prime ideals of R.

Motivation

In this talk, R is a commutative ring and M is a unital R-module.

Definition 1

Let R be a ring. An ideal P of R is **prime** if $rs \in P$ for $r, s \in R$ implies that $r \in P$ or $s \in P$.

Definition 2

The **nilradical** of R is the set $N(R) := \{r \in R : r^n = 0 \text{ for some } n \in \mathbb{N}\}$.

Theorem 3

The nilradical of R is equal to the intersection of all prime ideals of R.

Definition 4

A **radical** of an ideal I is the set $\sqrt{I} = \{r \in R : r^n \in I \text{ for some } n \in \mathbb{N}\}$.

Motivation

In this talk, R is a commutative ring and M is a unital R-module.

Definition 1

Let R be a ring. An ideal P of R is **prime** if $rs \in P$ for $r, s \in R$ implies that $r \in P$ or $s \in P$.

Definition 2

The **nilradical** of R is the set $N(R) := \{r \in R : r^n = 0 \text{ for some } n \in \mathbb{N}\}$.

Theorem 3

The nilradical of R is equal to the intersection of all prime ideals of R.

Definition 4

A **radical** of an ideal I is the set $\sqrt{I} = \{r \in R : r^n \in I \text{ for some } n \in \mathbb{N}\}$.

Corollary 5

The radical of an ideal I of R is equal to the intersection of all prime ideals of R containing I.

Definition 6 ([Dauns, 1978])

A submodule P of M is **prime** if for every $r \in R, m \in M$, $rm \in P$ implies that $m \in P$ or $r \in (P : M)$ where
$(P : M) = \mathrm{ann_R(M/P) = \{r \in R : rM \subseteq P\}}$.
The ideal $(P : M)$ is called the **residual ideal** of P in M.

Radical of a Submodule

Definition 7

A **classical radical** of a submodule $N \leq M$ is the radical of $\mathrm{ann_R(M/N)}$.

Radical of a Submodule

Definition 7

A **classical radical** of a submodule $N \leq M$ is the radical of $\mathrm{ann_R(M/N)}$.

Example 8 (McCasland and Moore, 1986)

Let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}$. Consider $0 \neq N \leq M$ where $N = R(x, 0)$ and $x \in \mathbb{Z}$, Then $(N : M) = \mathrm{ann_R(M/N)} = 0$ and $\sqrt{(N : M)} = 0$.

Radical of a Submodule

Definition 7

A **classical radical** of a submodule $N \leq M$ is the radical of $\mathrm{ann_R(M/N)}$.

Example 8 (McCasland and Moore, 1986)

Let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}$. Consider $0 \neq N \leq M$ where $N = R(x, 0)$ and $x \in \mathbb{Z}$, Then $(N : M) = \mathrm{ann_R(M/N)} = 0$ and $\sqrt{(N : M)} = 0$.

Definition 9 ([McCasland and Moore, 1986])

An (M-)**radical** of a submodule N of M, denoted by $\mathrm{rad}_M(N)$, is an intersection of all prime submodules of M containing N. If there is no prime submodule containing N, then $\mathrm{rad}_M(N) = M$.

Definition 10 ([McCasland and Moore, 1991])

An **envelope** of a submodule N of M is the set $E_M(N) := \{x \in M : x = rm, r^k m \in N \text{ for } r \in R, m \in M, k \in \mathbb{N}\}$.

Radical Formula of Modules

Definition 10 ([McCasland and Moore, 1991])

An **envelope** of a submodule N of M is the set
$E_M(N) := \{x \in M : x = rm, r^k m \in N \text{ for } r \in R, m \in M, k \in \mathbb{N}\}$.

Definition 11 ([McCasland and Moore, 1991])

An R-module M is said **satisfies the radical formula (s.t.r.f)** if $\mathrm{rad}_M(N) = R(E_M(N))$ for every submodule N of M.
A ring R is **satisfies the radical formula (s.t.r.f)** if every R-module s.t.r.f.

Why?

Why? (1/3)

Theorem 12

- *(McCasland and Moore, 1991) Any finitely generated module over a PID s.t.r.f.*
- *(Jenkins and Smith, 1992) Any Dedekind domain s.t.r.f.*
- *(Sharif, Sharifi, and Namazi, 1996) Any Artinian ring s.t.r.f.*
- *(Sharif, Sharifi, and Namazi, 1996) Any ZPI-ring R (Every ideal of R can be written as product of prime ideals) s.t.r.f.*
- *(Sharif, Sharifi, and Namazi, 1996) Finite direct sum of Dedekind domain s.t.r.f.*
- *(Leung and Man, 1997) Almost multiplication ring R (Every ideal $I \lhd R$ such that $\sqrt{I}$ is prime, $I = (\sqrt{I})^n$) s.t.r.f.*
- *(Alkan, 2007) A ring R such that $R/\mathrm{rad}(R)$ semisimple s.t.r.f.*
- *(Parkash, 2012) Any Arithmetical ring R (For all ideals $I, J, K \lhd R$, $I \cap (J + K) = I \cap J + I \cap K$) s.t.r.f.*

Why? (2/3)

The radical formula characterise rings.

Theorem 13 (Jenkins and Smith, 1992)

A Noetherian UFD R is a PID if and only if R s.t.r.f.

Theorem 14 (Man, 1996)

Let R be a Noetherian domain of Krull dimension 1. If R-module $R \oplus R$ s.t.r.f. then R is a Dedekind domain.

Why? (3/3)

Interesting characterisations.

Theorem 15 ([McCasland and Moore, 1991])

The following statements are equivalent for a ring R.

1. *R s.t.r.f.*
2. *Every free module over R s.t.r.f.*
3. *Every faithful module over R s.t.r.f.*
4. *R is a homomorphic image of a ring that s.t.r.f.*

The Generalized Radical Formula

Goal

One of the Goals of this works (not this talk) is to characterise the classes of rings with another version of radical formula by utilizing the weakly prime notions.

The Generalized Radical Formula

Goal

One of the Goals of this works (not this talk) is to characterise the classes of rings with another version of radical formula by utilizing the weakly prime notions.

This talk

We study the classes of the modules in which we can "transfer" the radical formula. For instance, in some situation the problem is simplified for the class of rings that s.t.r.f.

Weakly prime version of radical and envelope

Definition 16 ([Atani and Farzalipour, 2007])

A submodule P of M is **weakly prime** if for every $r \in R, m \in M$, $0 \neq rm \in P$ implies that $r \in (P : M)$ or $m \in P$.

Weakly prime version of radical and envelope

Definition 16 ([Atani and Farzalipour, 2007])

A submodule P of M is **weakly prime** if for every $r \in R, m \in M$, $0 \neq rm \in P$ implies that $r \in (P : M)$ or $m \in P$.

Definition 17

Let N be a submodule of R-module M and $n \in \mathbb{N}$.

- **Weakly prime radical** of a submodule N of M denoted by $\mathrm{W\text{-}rad}_M(N)$ is defined as intersection of all weakly prime submodules of M containing N. If there is no prime submodule containing N, then $\mathrm{W\text{-}rad}_M(N) = M$.
- **Weakly envelope** of N in M is defined to be the set $\mathrm{WE}_M(N) = \{x \in M : x = rm, r \in R, m \in M$ such that $0 \neq r^k m \in N$ for some $k \in \mathbb{N}\}$.

Almost Prime Submodules

Definition 18 ([Bhatwadekars and Sharma, 2005])

A proper ideal I is called **almost prime** if for all $r, s \in R$ with $rs \in I \setminus I^2$, either $r \in I$ or $s \in I$.

Almost Prime Submodules

Definition 18 ([Bhatwadekars and Sharma, 2005])

A proper ideal I is called **almost prime** if for all $r, s \in R$ with $rs \in I \setminus I^2$, either $r \in I$ or $s \in I$.

Definition 19 ([Moradi and Azizi, 2013])

For a positive integer $n \geq 2$, a proper submodule N of M is called an n-**almost prime** submodule if for any $r \in R$, $m \in M$ such that $rm \in N \setminus (N : M)^{n-1}N$, either $m \in N$ or $r \in (N : M)$.

The Radical Formulas

In this talk, we associate 0 to prime submodules and 1 to weakly prime submodules.

Definition 20

Let $e, r \in \{0, 1\}$. An R-module M is said to satisfies the (e, r) radical formula (s.t.(e, r).r.f) if $R\,(e\text{-}E_M(N)) = r\text{-}rad_M(N)$ for every submodule N of M.

The Radical Formulas

In this talk, we associate 0 to prime submodules and 1 to weakly prime submodules.

Definition 20

Let $e, r \in \{0, 1\}$. An R-module M is said to satisfies the (e, r) radical formula (s.t.(e, r).r.f) if $R\left(e\text{-}E_M(N)\right) = r\text{-}rad_M(N)$ for every submodule N of M.

$$\begin{array}{ccc} R\left(WE_M(N)\right) & \subseteq & R\left(E_M(N)\right) \\ \cap\!| & & \cap\!| \\ W\text{-}rad_M(N) & \subseteq & rad_M(N) \end{array}$$

Radical Formula and Localization

Theorem 21 (S., 2024)

A module M s.t.(e,0).r.f. for $e \in \{0,1\}$ if and only if the $R_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ satisfies the same radical formula for every maximal ideals $\mathfrak{m}$ of R.

Problem with Weakly Prime Radical

Lemma 22

Let P be a submodule of M. If P is weakly prime, then $P_{\mathfrak{m}}$ is weakly prime $R_{\mathfrak{m}}$-submodule of $M_{\mathfrak{m}}$.

Problem with Weakly Prime Radical

Lemma 22

Let P be a submodule of M. If P is weakly prime, then $P_{\mathfrak{m}}$ is weakly prime $R_{\mathfrak{m}}$-submodule of $M_{\mathfrak{m}}$.

The converse is not necessarily true

Let $\mathfrak{m}$ be a maximal ideal of R and N be an $R_{\mathfrak{m}}$-submodule of $M_{\mathfrak{m}}$. The preimage of N is not necessarily weakly prime.

Problem with Weakly Prime Radical

Definition 23 ([Dauns, 1977])

A submodule $N \leq M$ is called **primal** if the set $S(N) = \{r \in R : \exists m \notin N \text{ such that } rm \in N\}$ forms an ideal of R.

Problem with Weakly Prime Radical

Definition 23 ([Dauns, 1977])

A submodule $N \leq M$ is called **primal** if the set $S(N) = \{r \in R : \exists m \notin N \text{ such that } rm \in N\}$ forms an ideal of R.

Lemma 24

Let M be an R-module. There is a one-to-one correspondence between weakly prime submodule P of M with $(R \setminus \mathfrak{m}) \cap (P : M) = \emptyset$ and weakly prime $R_\mathfrak{m}$-submodule $P_\mathfrak{m}$ of $M_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R; if it satisfies one of the following statements:

1. *$S(0_M) \subseteq J(R)$, where $J(R)$ is the Jacobson radical of R.*
2. *For every maximal ideal $\mathfrak{m}$ of R and every submodule P of M; if $P_\mathfrak{m}$ is weakly prime, then P is a primal submodule.*

Problem with Weakly Prime Radical

Definition 23 ([Dauns, 1977])

A submodule $N \leq M$ is called **primal** if the set $S(N) = \{r \in R : \exists m \notin N \text{ such that } rm \in N\}$ forms an ideal of R.

Lemma 24

Let M be an R-module. There is a one-to-one correspondence between weakly prime submodule P of M with $(R \setminus \mathfrak{m}) \cap (P : M) = \emptyset$ and weakly prime $R_\mathfrak{m}$-submodule $P_\mathfrak{m}$ of $M_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R; if it satisfies one of the following statements:

1. *$S(0_M) \subseteq J(R)$, where $J(R)$ is the Jacobson radical of R.*
2. *For every maximal ideal $\mathfrak{m}$ of R and every submodule P of M; if $P_\mathfrak{m}$ is weakly prime, then P is a primal submodule.*

Definition 25

A module satisfies the second condition of the Lemma 24 is said to be **lwpp**.

Theorem 26 (S., 2024)

Let M be an R-module satisfies one of the following properties:

1. $S(0_M) \subseteq J(R)$.
2. *M is lwpp.*

Then M s.t.(e,1).r.f. for $e \in \{0, 1\}$ if and only if the $R_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ satisfies the same radical formula for every maximal ideals $\mathfrak{m}$ of R.

Radical Formula and Direct Summand

Definition 27 (Alsuraiheed and Bavula, 2021)

An **intersection condition** holds for a direct sum $M = \oplus_{\lambda\in\Lambda} M_\lambda$ of non-zero R-modules M_λ if for every submodule N of M,

$$N = \bigoplus_{\lambda\in\Lambda} (N \cap M_\lambda)$$

Radical Formula and Direct Summand

Definition 27 (Alsuraiheed and Bavula, 2021)

An **intersection condition** holds for a direct sum $M = \oplus_{\lambda \in \Lambda} M_\lambda$ of non-zero R-modules M_λ if for every submodule N of M,

$$N = \bigoplus_{\lambda \in \Lambda} (N \cap M_\lambda)$$

Proposition 28 (S.,2024)

Let $M = M_1 \oplus M_2$ be an R-module. If the intersection condition holds for the direct sum M, then the following statements are true:

1. *M s.t.r.f. if and only if M_1 and M_2 s.t.r.f.*
2. *If M is a semiprime module, then M s.t.(1,r).r.f for $r = 0, 1$ if and only if M_1 and M_2 s.t.(1,r).r.f.*

Definition 29 (Faith, 1973)

An R-module M is **divisible** if for every $m \in M$ and $r \in R$ there exists $m' \in M$ such that $m = rm'$.

Radical Formula and Direct Summand

Definition 29 (Faith, 1973)

An R-module M is **divisible** if for every $m \in M$ and $r \in R$ there exists $m' \in M$ such that $m = rm'$.

Proposition 30 (S., 2024)

Let R be a domain. Let $M = M_1 \oplus M_2$ be a direct sum of a divisible module M_1 and a module M_2.

1. *If M_2 s.t.$(0,r)$.r.f, then M s.t.$(0,r)$.r.f.*
2. *If M_2 s.t.$(1,r)$.r.f and M is torsion-free, then M s.t.$(1,r)$.r.f.*

Proposition 31 (S., 2024)

Let N be a pure weakly semiprime submodule of M.

1. *If M s.t.(1,1).r.f., then N s.t.(1,1).r.f.*
2. *If M s.t.(1,0).r.f., then N s.t.(1,0).r.f.*

Radical Formula and Submodules

Proposition 31 (S., 2024)

Let N be a pure weakly semiprime submodule of M.

1. *If M s.t.(1,1).r.f., then N s.t.(1,1).r.f.*
2. *If M s.t.(1,0).r.f., then N s.t.(1,0).r.f.*

Definition 32 (Farzalipour, 2014)

A submodule P of M is **weakly semiprime** if $0 \neq r^k m \in P$ for $r \in R, m \in M, k \in \mathbb{N}$ implies that $rm \in P$.

Proposition 31 (S., 2024)

Let N be a pure weakly semiprime submodule of M.

1. *If M s.t.(1,1).r.f., then N s.t.(1,1).r.f.*
2. *If M s.t.(1,0).r.f., then N s.t.(1,0).r.f.*

Definition 32 (Farzalipour, 2014)

A submodule P of M is **weakly semiprime** if $0 \neq r^k m \in P$ for $r \in R, m \in M, k \in \mathbb{N}$ implies that $rm \in P$.

Definition 33 (Anderson and Fuller, 1992)

A submodule N of M is **pure** if $IM \cap N = IN$ for any ideal I of R. A module M is **regular** if every submodule of M is pure.

Radical Formula and Homomorphic Images

Theorem 34 (McCasland and Moore, 1991)

Any homomorphic image of a module that s.t.r.f. also s.t.r.f.

Proposition 35 (S., 2024)

Let M be an R-module and $r \in \{0, 1\}$.

1. *If K is a semiprime submodule of M (e.g. M is a regular semiprime module) and M s.t.(1,r).r.f for $r = 0, 1$, then M/K also s.t.(1,r).r.f.*
2. *If M s.t.(0,1).r.f., $K \leq M$ and there is a one-to-one correspondence between weakly prime submodules of M/K and weakly prime submodules of M containing K, then M/K s.t.(0,1).r.f.*

Thank You

M. Alkan and Y. Tiras, On prime submodules, *Rocky Mountain Journal of Mathematics*, **37**(6) (2007).

D.D. Anderson and M. Bataineh, Generalization of prime ideals, *Communications in Algebra*, **36** (2008), 785-696.

S.E. Atani and F. Farzalipour, On weakly prime submodules, *Tamkang Journal of Mathematics*, **38**(3) (2007), 247-252.

S.E. Atani and F.E.K. Saraei, On McCasland submodules, *International Mathematical Forum*, **2**(46) (2007), 2255-2260.

S. M. Bhatwadekar and P. K. Sharma, Unique factorization and birth of almost primes, *Communications in Algebra*, **33** (2005), 43-49.

J. Dauns, Primal modules, *Communications in Algebra*, **25** (1997), 2409-2435.

J. Dauns, Prime modules, *Journal für die reine und angewandte Mathematik*, **298** (1978), 156-181.

J. Jenkins and P.F. Smith, On the prime radical of a module over a commutative ring, *Communications in Algebra*, **20**(12) (1992), 3593-3602.

K.H. Leung and S.H. Man, On commutative Noetherian rings which satisfy the ,radical formula. *Glasgow Mathematical Journal* **39**(3) (1997), 285-293.

R.L. McCasland and M.E. Moore, On radicals of submodules of finitely generated modules, *Canad. Math. Bull.*, **29**(1) (1986), 37-39.

R.L. McCasland and M.E. Moore, On radicals of submodules, *Communications in Algebra*, **19**(5) (1991), 1327-1341.

A. Parkash, Arithmetical ring satisfy the radical formula, *Journal of Commutative Algebra*, **4**(2) (2012), 293-296.

S. Moradi and A. Azizi, n-almost prime submodules, *Indian J. Pure Appl.. Math*, **44**(5) (2013), 605-619.

H. Sharif, Y. Sharifi, and S. Namazi, Ring satisfying the radical formula, *Acta Math Hung*, **71** (1996), 103-108.

Steven, The relation between generalized notions of radical formula of a module, *Communications in Algebra*, **52**(3)(2024), 1179-1195.