The Covering Numbers of Rings I

Eric Swartz
(joint with Nicholas Werner)

William & Mary

July 14, 2025

Starting question

Question (1969 Putnam Competition, B2)

G is a finite group with identity 1. Show that we cannot find two proper subgroups A and $B(\neq \{1\}$ or $G)$ such that $A \cup B = G$. Can we find three proper subgroups A, B, C such that $A \cup B \cup C = G$?

Early history: groups

Definition

- A cover of a group G is a collection of proper subgroups of G whose set theoretic union is all of G.
- Assuming a cover exists for G, the covering number $\sigma(G)$ of G is the size of a minimum cover.

Early history: groups

Definition

- A cover of a group G is a collection of proper subgroups of G whose set theoretic union is all of G.
- Assuming a cover exists for G, the covering number $\sigma(G)$ of G is the size of a minimum cover.

Theorem (Scorza [Sco26])

A group G has $\sigma(G) = 3$ if and only if there is a surjective homomorphism from G onto the Klein 4-group, $C_2 \times C_2$.

This result was "rediscovered" many times in subsequent years!

For a nice proof of Scorza's result, see Bhargava [Bha09].

Motivating question: What about other integers???

Given an integer n, is there a group G with $\sigma(G) = n$?

Motivating question: What about other integers???

Given an integer n, is there a group G with $\sigma(G) = n$?

Known Results

- Cohn [Coh94]: $p^d + 1$ is a covering number for p prime
- Tomkinson [Tom97]: there is no group G such that $\sigma(G) = 7$
- Detomi, Lucchini [DL08]: there is no group G such that $\sigma(G) = 11$
- Garonzi [Gar13]: classified integers $\leqslant 25$ that are covering numbers
- Garonzi, Kappe, S. [GKS22]: classified integers $\leqslant 129$ that are covering numbers

Motivating question: What about other integers???

Given an integer n, is there a group G with $\sigma(G) = n$?

Known Results

- Cohn [Coh94]: $p^d + 1$ is a covering number for p prime
- Tomkinson [Tom97]: there is no group G such that $\sigma(G) = 7$
- Detomi, Lucchini [DL08]: there is no group G such that $\sigma(G) = 11$
- Garonzi [Gar13]: classified integers $\leqslant 25$ that are covering numbers
- Garonzi, Kappe, S. [GKS22]: classified integers $\leqslant 129$ that are covering numbers

Not covering numbers: 2, 7, 11

Motivating question: What about other integers???

Given an integer n, is there a group G with $\sigma(G) = n$?

Known Results

- Cohn [Coh94]: $p^d + 1$ is a covering number for p prime
- Tomkinson [Tom97]: there is no group G such that $\sigma(G) = 7$
- Detomi, Lucchini [DL08]: there is no group G such that $\sigma(G) = 11$
- Garonzi [Gar13]: classified integers $\leqslant 25$ that are covering numbers
- Garonzi, Kappe, S. [GKS22]: classified integers $\leqslant 129$ that are covering numbers

Not covering numbers: 2, 7, 11, 19, 22, 25

Motivating question: What about other integers???

Given an integer n, is there a group G with $\sigma(G) = n$?

Known Results

- Cohn [Coh94]: $p^d + 1$ is a covering number for p prime
- Tomkinson [Tom97]: there is no group G such that $\sigma(G) = 7$
- Detomi, Lucchini [DL08]: there is no group G such that $\sigma(G) = 11$
- Garonzi [Gar13]: classified integers $\leqslant 25$ that are covering numbers
- Garonzi, Kappe, S. [GKS22]: classified integers $\leqslant 129$ that are covering numbers

Not covering numbers: 2, 7, 11, 19, 22, 25, 27, 34, 35, 37, 39, 41, 43, 45, 47, 49, 51, 52, 53, 55, 56, 58, 59, 61, 66, 69, 70, 75, 76, 77, 78, 79, 81, 83, 87, 88, 89, 91, 93, 94, 95, 96, 97, 99, 100, 101, 103, 105, 106, 107, 109, 111, 112, 113, 115, 116, 117, 118, 119, 120, 123, 124, 125

A conjecture for groups...

Conjecture (Garonzi, Kappe, S. [GKS22])

There are infinitely many integers that are not covering numbers of groups. Moreover,

$$\lim_{N\to\infty} \frac{\#\ \textit{of integers} \leqslant N\ \textit{that are covering numbers}}{N} = 0$$

. . . and some results for other structures

- cover of an algebraic structure A: proper algebraic substructures whose set theoretic union is A
- covering number: size of a minimum cover

Theorem (Gagola III, Kappe [GK16])

Every integer $n > 2$ is a covering number of a loop.

Theorem (Donoven, Kappe [DK23])

- *The covering number of a finite semigroup that is not a group and not generated by a single element is always two.*
- *For each $n \geqslant 2$, there exists an inverse semigroup whose covering number (by inverse subsemigroups) is exactly n.*

What about rings???

Definition

A ring is a set R equipped with binary operations $+$ and $\cdot$ satisfying:

- $(R, +)$ is an abelian group,
- multiplication is associative,
- distributive laws hold.

Definition

A ring with unity is a ring that also has a unity (multiplicative identity).

NOTE: some authors refer to these as "rngs" and "rings," respectively

The covering number of a ring

Definition

For us, a subring $S \subseteq R$ is a group under addition and closed under multiplication; that is, **a subring need not contain a multiplicative identity**.

Definition

- A cover of a ring R is a collection of proper subrings of R whose set theoretic union is all of R.
- Assuming a cover exists for R, the covering number $\sigma(R)$ of R is the size of a minimum cover.

Which integers are covering numbers of rings?

How does one even begin to answer this question???

Lemma

If I is an ideal of a ring R and R/I admits a finite cover, then $\sigma(R) \leqslant \sigma(R/I)$.

IDEA: Take the inverse image under the natural projection of a cover of R/I!

σ-elementary rings

Definition

A ring R is σ-elementary if $\sigma(R) < \sigma(R/I)$ for every nonzero two-sided ideal I of R.

σ-elementary rings

Definition

A ring R is σ-elementary if $\sigma(R) < \sigma(R/I)$ for every nonzero two-sided ideal I of R.

Theorem (B. H. Neumann [Neu54], Lewin [Lew67])

If ring R has finite covering number, then there exists a finite homomorphic image of R with the same covering number.

In other words: to determine which integers are covering numbers of rings, it suffices to consider finite (σ-elementary) rings.

σ-elementary rings

Definition

A ring R is σ-elementary if $\sigma(R) < \sigma(R/I)$ for every nonzero two-sided ideal I of R.

Theorem (B. H. Neumann [Neu54], Lewin [Lew67])

If ring R has finite covering number, then there exists a finite homomorphic image of R with the same covering number.

In other words: to determine which integers are covering numbers of rings, it suffices to consider finite (σ-elementary) rings.

IDEA: Classify all σ-elementary rings!

Reductions

Lemma (S., Werner [SW21])

Let R be a ring with unity such that $\sigma(R)$ is finite. Then, there exists a two-sided ideal I of R such that:

- *R/I is finite;*
- *R/I has characteristic p;*
- *$\mathscr{J}(R/I)^2 = \{0\}$;*
- *and $\sigma(R/I) = \sigma(R)$.*

The Wedderburn-Malcev Theorem

Theorem (Wedderburn-Malcev Theorem)

R: finite ring with unity of characteristic p
Then, there exists an $\mathbb{F}_p$-subalgebra S of R such that:

- $R = S \oplus \mathscr{J}(R)$,
- $S \cong R/\mathscr{J}(R)$ *as* $\mathbb{F}_p$*-algebras,*
- S*: unique up to conjugation by elements of* $1 + \mathscr{J}(R)$.

- $\mathscr{J}(R)$: Jacobson radical
- S: semisimple
- $S \cong S_1 \oplus \cdots \oplus S_t$, each S_i simple
- So, each $S_i \cong \begin{cases} \text{finite field } \mathbb{F}_{p^{d_i}} \\ \text{full matrix ring } M_{n_i}(p^{d_i}) \end{cases}$

Example: R semisimple, R/J commutative (Products of fields)

R	Coverable? $\sigma(R)$?
$\mathbb{F}_2$	

Example: R semisimple, R/J commutative (Products of fields)

R	Coverable? $\sigma(R)$?
$\mathbb{F}_2$	Not coverable!

Example: R semisimple, R/J commutative (Products of fields)

R	Coverable? $\sigma(R)$?
$\mathbb{F}_2$	Not coverable!
$\mathbb{F}_2 \oplus \mathbb{F}_2$	

Example: R semisimple, R/J commutative (Products of fields)

R	Coverable? $\sigma(R)$?
$\mathbb{F}_2$	Not coverable!
$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\sigma(R) = 3$

Example: R semisimple, R/J commutative (Products of fields)

R	Coverable? $\sigma(R)$?
$\mathbb{F}_2$	Not coverable!
$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\sigma(R) = 3$
$\mathbb{F}_3 \oplus \mathbb{F}_3$	

Example: R semisimple, R/J commutative (Products of fields)

R	Coverable? $\sigma(R)$?
$\mathbb{F}_2$	Not coverable!
$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\sigma(R) = 3$
$\mathbb{F}_3 \oplus \mathbb{F}_3$	Not coverable! $R = \langle (1, -1) \rangle$

Example: R semisimple, R/J commutative (Products of fields)

R	Coverable? $\sigma(R)$?
$\mathbb{F}_2$	Not coverable!
$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\sigma(R) = 3$
$\mathbb{F}_3 \oplus \mathbb{F}_3$	Not coverable! $R = \langle (1, -1) \rangle$
$\mathbb{F}_3 \oplus \mathbb{F}_3 \oplus \mathbb{F}_3$	

Example: R semisimple, R/J commutative (Products of fields)

R	Coverable? $\sigma(R)$?
$\mathbb{F}_2$	Not coverable!
$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\sigma(R) = 3$
$\mathbb{F}_3 \oplus \mathbb{F}_3$	Not coverable! $R = \langle (1, -1) \rangle$
$\mathbb{F}_3 \oplus \mathbb{F}_3 \oplus \mathbb{F}_3$	$\sigma(R) = 6$

Example: R semisimple, R/J commutative (Products of fields)

R	Coverable? $\sigma(R)$?
$\mathbb{F}_2$	Not coverable!
$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\sigma(R) = 3$
$\mathbb{F}_3 \oplus \mathbb{F}_3$	Not coverable! $R = \langle (1, -1) \rangle$
$\mathbb{F}_3 \oplus \mathbb{F}_3 \oplus \mathbb{F}_3$	$\sigma(R) = 6$
$\mathbb{F}_4 \oplus \mathbb{F}_4 \oplus \mathbb{F}_4 \oplus \mathbb{F}_4$	

Example: R semisimple, R/J commutative (Products of fields)

R	Coverable? $\sigma(R)$?
$\mathbb{F}_2$	Not coverable!
$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\sigma(R) = 3$
$\mathbb{F}_3 \oplus \mathbb{F}_3$	Not coverable! $R = \langle (1, -1) \rangle$
$\mathbb{F}_3 \oplus \mathbb{F}_3 \oplus \mathbb{F}_3$	$\sigma(R) = 6$
$\mathbb{F}_4 \oplus \mathbb{F}_4 \oplus \mathbb{F}_4 \oplus \mathbb{F}_4$	$\sigma(R) = 4$

Example: R semisimple, R/J commutative (Products of fields)

R	Coverable? $\sigma(R)$?
$\mathbb{F}_2$	Not coverable!
$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\sigma(R) = 3$
$\mathbb{F}_3 \oplus \mathbb{F}_3$	Not coverable! $R = \langle (1, -1) \rangle$
$\mathbb{F}_3 \oplus \mathbb{F}_3 \oplus \mathbb{F}_3$	$\sigma(R) = 6$
$\mathbb{F}_4 \oplus \mathbb{F}_4 \oplus \mathbb{F}_4 \oplus \mathbb{F}_4$	$\sigma(R) = 4$
$\mathbb{F}_4 \oplus \mathbb{F}_4$	

Example: R semisimple, R/J commutative (Products of fields)

R	Coverable? $\sigma(R)$?
$\mathbb{F}_2$	Not coverable!
$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\sigma(R) = 3$
$\mathbb{F}_3 \oplus \mathbb{F}_3$	Not coverable! $R = \langle (1, -1) \rangle$
$\mathbb{F}_3 \oplus \mathbb{F}_3 \oplus \mathbb{F}_3$	$\sigma(R) = 6$
$\mathbb{F}_4 \oplus \mathbb{F}_4 \oplus \mathbb{F}_4 \oplus \mathbb{F}_4$	$\sigma(R) = 4$
$\mathbb{F}_4 \oplus \mathbb{F}_4$	$\sigma(R) = 4$

Example: R semisimple, R/J commutative (Products of fields)

R	Coverable? $\sigma(R)$?
$\mathbb{F}_2$	Not coverable!
$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\sigma(R) = 3$
$\mathbb{F}_3 \oplus \mathbb{F}_3$	Not coverable! $R = \langle (1, -1) \rangle$
$\mathbb{F}_3 \oplus \mathbb{F}_3 \oplus \mathbb{F}_3$	$\sigma(R) = 6$
$\mathbb{F}_4 \oplus \mathbb{F}_4 \oplus \mathbb{F}_4 \oplus \mathbb{F}_4$	$\sigma(R) = 4$
$\mathbb{F}_4 \oplus \mathbb{F}_4$	$\sigma(R) = 4$

What's going on???

Example: R semisimple, R/J commutative, cont.

(Werner [Wer15])

- $q = p^d$, a prime power

Example: R semisimple, R/J commutative, cont.

(Werner [Wer15])

- $q = p^d$, a prime power
- $\tau(q) := \begin{cases} p, \text{ if } d = 1, \\ |\{f \in \mathbb{F}_p[x] : \deg(f) = d, f \text{ monic, irred.}\}| + 1, \text{ if } d > 1 \end{cases}$

Example: R semisimple, R/J commutative, cont.

(Werner [Wer15])

- $q = p^d$, a prime power
- $\tau(q) := \begin{cases} p, \text{ if } d = 1, \\ |\{f \in \mathbb{F}_p[x] : \deg(f) = d, f \text{ monic, irred.}\}| + 1, \text{ if } d > 1 \end{cases}$
- $\nu(q) := \begin{cases} 1, \text{ if } d = 1, \\ \# \text{ distinct prime factors of d}, \text{ if } d > 1 \end{cases}$

Example: R semisimple, R/J commutative, cont.

(Werner [Wer15])

- $q = p^d$, a prime power
- $\tau(q) := \begin{cases} p, & \text{if } d = 1, \\ |\{f \in \mathbb{F}_p[x] : \deg(f) = d, f \text{ monic, irred.}\}| + 1, & \text{if } d > 1 \end{cases}$
- $\nu(q) := \begin{cases} 1, & \text{if } d = 1, \\ \# \text{ distinct prime factors of d}, & \text{if } d > 1 \end{cases}$
- $R = \bigoplus_{i=1}^{t} \mathbb{F}_q$ is coverable iff $t \geqslant \tau(q)$

Example: R semisimple, R/J commutative, cont.

(Werner [Wer15])

- $q = p^d$, a prime power
- $\tau(q) := \begin{cases} p, \text{ if } d = 1, \\ |\{f \in \mathbb{F}_p[x] : \deg(f) = d, f \text{ monic, irred.}\}| + 1, \text{ if } d > 1 \end{cases}$
- $\nu(q) := \begin{cases} 1, \text{ if } d = 1, \\ \# \text{ distinct prime factors of d}, \text{ if } d > 1 \end{cases}$
- $R = \bigoplus_{i=1}^{t} \mathbb{F}_q$ is coverable iff $t \geqslant \tau(q)$

When R is coverable,

$$\sigma(R) = \tau(q)\nu(q) + d\binom{\tau(q)}{2}.$$

Example: R semisimple, R/J noncommutative (Full matrix rings)

(Lucchini, Maróti [LM10], Crestani [Cre12])

- $R = M_n(q)$, $n \geqslant 2$ (all $n \times n$ matrices with entries in $\mathbb{F}_q$)

Example: R semisimple, R/J noncommutative (Full matrix rings)

(Lucchini, Maróti [LM10], Crestani [Cre12])

- $R = M_n(q)$, $n \geqslant 2$ (all $n \times n$ matrices with entries in $\mathbb{F}_q$)
- a: smallest prime divisor of n
- $\binom{n}{k}_q := \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-(k-1)}-1}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}$

Example: R semisimple, R/J noncommutative (Full matrix rings)

(Lucchini, Maróti [LM10], Crestani [Cre12])

- $R = M_n(q)$, $n \geqslant 2$ (all $n \times n$ matrices with entries in $\mathbb{F}_q$)
- a: smallest prime divisor of n
- $\binom{n}{k}_q := \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-(k-1)}-1}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}$

$$\sigma(R) = \frac{1}{a} \prod_{k=1;a\nmid k}^{n-1} (q^n - q^k) + \sum_{k=1;a\nmid k}^{\lfloor n/2 \rfloor} \binom{n}{k}_q$$

Example: R semisimple, R/J noncommutative (Full matrix rings)

(Lucchini, Maróti [LM10], Crestani [Cre12])

- $R = M_n(q)$, $n \geqslant 2$ (all $n \times n$ matrices with entries in $\mathbb{F}_q$)
- a: smallest prime divisor of n
- $\binom{n}{k}_q := \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-(k-1)}-1}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}$

$$\sigma(R) = \frac{1}{a}\prod_{k=1;a\nmid k}^{n-1}(q^n - q^k) + \sum_{k=1;a\nmid k}^{\lfloor n/2 \rfloor}\binom{n}{k}_q$$

- Peruginelli, Werner [PW18]: the only noncommutative semisimple σ-elementary rings are $M_n(q)$

(S., Werner [SW21])

- $q = p^d$
- $R = \mathbb{F}_q(+)\mathbb{F}_q^2$, idealization of $\mathbb{F}_q$ with 2-dimensional vector space $\mathbb{F}_q^2$

Example: R not semisimple, R/J commutative

(S., Werner [SW21])

- $q = p^d$
- $R = \mathbb{F}_q(+)\mathbb{F}_q^2$, idealization of $\mathbb{F}_q$ with 2-dimensional vector space $\mathbb{F}_q^2$
- $R := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : a, b, c \in \mathbb{F}_q \right\}$

Example: R not semisimple, R/J commutative

(S., Werner [SW21])

- $q = p^d$
- $R = \mathbb{F}_q(+)\mathbb{F}_q^2$, idealization of $\mathbb{F}_q$ with 2-dimensional vector space $\mathbb{F}_q^2$
- $R := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : a, b, c \in \mathbb{F}_q \right\}$
- Here: S, J

Example: R not semisimple, R/J commutative

(S., Werner [SW21])

- $q = p^d$
- $R = \mathbb{F}_q(+)\mathbb{F}_q^2$, idealization of $\mathbb{F}_q$ with 2-dimensional vector space $\mathbb{F}_q^2$
- $R := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : a, b, c \in \mathbb{F}_q \right\}$
- Here: S, J

$$\sigma(R) = q + 1$$

R not semisimple, R/J noncommutative???

- $R := \left(\begin{array}{c|c} M_4(2) & M_{4\times 1}(2) \\ \hline 0 & \mathbb{F}_2 \end{array}\right)$

R not semisimple, R/J noncommutative???

- $R := \left(\begin{array}{c|c} M_4(2) & M_{4\times 1}(2) \\ \hline 0 & \mathbb{F}_2 \end{array}\right)$
- S, J

R not semisimple, R/J noncommutative???

- $R := \left(\begin{array}{c|c} M_4(2) & M_{4\times 1}(2) \\ \hline 0 & \mathbb{F}_2 \end{array}\right)$
- S, J
- There are $|J| = 2^4 = 16$ complements to J in R

R not semisimple, R/J noncommutative???

- $R := \left(\begin{array}{c|c} M_4(2) & M_{4\times 1}(2) \\ \hline 0 & \mathbb{F}_2 \end{array}\right)$
- S, J
- There are $|J| = 2^4 = 16$ complements to J in R
- For each 1-dimensional subspace U of $\mathbb{F}_2^4$, let $T_U \subseteq M_4(2)$ be the set of all elements stabilizing U.

R not semisimple, R/J noncommutative???

- $R := \left(\begin{array}{c|c} M_4(2) & M_{4\times 1}(2) \\ \hline 0 & \mathbb{F}_2 \end{array}\right)$
- S, J
- There are $|J| = 2^4 = 16$ complements to J in R
- For each 1-dimensional subspace U of $\mathbb{F}_2^4$, let $T_U \subseteq M_4(2)$ be the set of all elements stabilizing U.
- There are $\binom{4}{1}_2 = \frac{2^4-1}{2-1} = 15$ maximal subrings of the form

$$\left(\begin{array}{c|c} T_U & M_{4\times 1}(2) \\ \hline 0 & \mathbb{F}_2 \end{array}\right)$$

R not semisimple, R/J noncommutative???

- $R := \left(\begin{array}{c|c} M_4(2) & M_{4\times 1}(2) \\ \hline 0 & \mathbb{F}_2 \end{array}\right)$
- S, J
- There are $|J| = 2^4 = 16$ complements to J in R
- For each 1-dimensional subspace U of $\mathbb{F}_2^4$, let $T_U \subseteq M_4(2)$ be the set of all elements stabilizing U.
- There are $\binom{4}{1}_2 = \frac{2^4-1}{2-1} = 15$ maximal subrings of the form

$$\left(\begin{array}{c|c} T_U & M_{4\times 1}(2) \\ \hline 0 & \mathbb{F}_2 \end{array}\right)$$

- Collectively, these rings form a (minimal) cover of size 31.

R not semisimple, R/J noncommutative???

- $R := \left(\begin{array}{c|c} M_4(2) & M_{4\times 1}(2) \\ \hline 0 & \mathbb{F}_2 \end{array}\right)$
- S, J
- There are $|J| = 2^4 = 16$ complements to J in R
- For each 1-dimensional subspace U of $\mathbb{F}_2^4$, let $T_U \subseteq M_4(2)$ be the set of all elements stabilizing U.
- There are $\binom{4}{1}_2 = \frac{2^4-1}{2-1} = 15$ maximal subrings of the form

$$\left(\begin{array}{c|c} T_U & M_{4\times 1}(2) \\ \hline 0 & \mathbb{F}_2 \end{array}\right)$$

- Collectively, these rings form a (minimal) cover of size 31.
- $\sigma(M_4(2)) = 71$, so R is σ-elementary!

Rings of AGL-type

Definition (AGL-type ring)

- $q_1 = p^{d_1}, q_2 = p^{d_2}$
- $q = q_1 \otimes q_2 := p^{\mathrm{LCM}(d_1,d_2)} = q_1^d$
- $$A(n, q_1, q_2) := \left(\begin{array}{c|c} M_n(q_1) & M_{n\times 1}(q) \\ \hline 0 & \mathbb{F}_{q_2} \end{array}\right)$$

Rings of AGL-type

Definition (AGL-type ring)

- $q_1 = p^{d_1}, q_2 = p^{d_2}$
- $q = q_1 \otimes q_2 := p^{\mathrm{LCM}(d_1,d_2)} = q_1^d$
- $$A(n, q_1, q_2) := \left(\begin{array}{c|c} M_n(q_1) & M_{n\times 1}(q) \\ \hline 0 & \mathbb{F}_{q_2} \end{array} \right)$$

Why "AGL-type?" Inspired by representations of the affine general linear group $\mathrm{AGL}(n, q)$, where

$$\mathrm{AGL}(n, q) \cong \left(\begin{array}{c|c} \mathrm{GL}(n, q) & M_{n\times 1}(q) \\ \hline 0 & 1 \end{array} \right)$$

How do we cover $A(n, q_1, q_2)$?

As it turns out, in general a minimal cover of $R = A(n, q_1, q_2)$ consists of:

How do we cover $A(n, q_1, q_2)$?

As it turns out, in general a minimal cover of $R = A(n, q_1, q_2)$ consists of:

- All q^n complements to J in R.

How do we cover $A(n, q_1, q_2)$?

As it turns out, in general a minimal cover of $R = A(n, q_1, q_2)$ consists of:

- All q^n complements to J in R.
- All $\binom{n}{d}_{q_1}$ maximal subrings of the form

$$\left(\begin{array}{c|c} T_U & M_{n\times 1}(q) \\ \hline 0 & \mathbb{F}_{q_2} \end{array} \right),$$

where U is a d-dimensional subspace of of $\mathbb{F}_{q_1}^n$ and T_U is the set of all elements of $M_n(q_1)$ stabilizing U.

How do we cover $A(n, q_1, q_2)$?

As it turns out, in general a minimal cover of $R = A(n, q_1, q_2)$ consists of:

- All q^n complements to J in R.
- All $\binom{n}{d}_{q_1}$ maximal subrings of the form

$$\left(\begin{array}{c|c} T_U & M_{n\times 1}(q) \\ \hline 0 & \mathbb{F}_{q_2} \end{array}\right),$$

  where U is a d-dimensional subspace of of $\mathbb{F}_{q_1}^n$ and T_U is the set of all elements of $M_n(q_1)$ stabilizing U.
- All $\omega(d)$ maximal subrings of the form

$$\left(\begin{array}{c|c} M_n(q_1) & M_{n\times 1}(q) \\ \hline 0 & \mathbb{F}_r \end{array}\right),$$

  where $\mathbb{F}_r$ is a maximal subfield of $\mathbb{F}_{q_2}$ containing $\mathbb{F}_{q_1} \cap \mathbb{F}_{q_2}$, where

$$\omega(d) = \#\text{ distinct prime divisors of } d$$

The covering number of $A(n, q_1, q_2)$

Theorem (S., Werner [SW24])

$R \cong A(n, q_1, q_2)$, where $n \geqslant 1$, and let

- $q := q_1 \otimes q_2 = q_1^d$,
- *a be the smallest prime divisor of n (if $n \geqslant 2$).*

Then, R is σ-elementary if and only if one of the following holds:

1. *$n = 1$ and $(q_1, q_2) \neq (2, 2)$ or $(4, 4)$. In this case, $\sigma(R) = q + 1$.*
2. *$n \geqslant 3$, $d < n - (n/a)$, and $(n, q_1) \neq (3, 2)$. In this case,*

$$\sigma(R) = q^n + \binom{n}{d}_{q_1} + \omega(d).$$

Thank you!

References I

Mira Bhargava, *Groups as unions of proper subgroups*, Amer. Math. Monthly **116** (2009), no. 5, 413–422. MR 2510838

J. H. E. Cohn, *On n-sum groups*, Math. Scand. **75** (1994), no. 1, 44–58. MR 1308936

Eleonora Crestani, *Sets of elements that pairwise generate a matrix ring*, Comm. Algebra **40** (2012), no. 4, 1570–1575. MR 2913003

Casey Donoven and Luise-Charlotte Kappe, *Finite coverings of semigroups and related structures*, Int. J. Group Theory **12** (2023), no. 3, 205–222. MR 4509642

Eloisa Detomi and Andrea Lucchini, *On the structure of primitive n-sum groups*, Cubo **10** (2008), no. 3, 195–210. MR 2467921

References II

- Martino Garonzi, *Finite groups that are the union of at most 25 proper subgroups*, J. Algebra Appl. **12** (2013), no. 4, 1350002, 11. MR 3037278
- Stephen M. Gagola, III and Luise-Charlotte Kappe, *On the covering number of loops*, Expo. Math. **34** (2016), no. 4, 436–447. MR 3578007
- Martino Garonzi, Luise-Charlotte Kappe, and Eric Swartz, *On integers that are covering numbers of groups*, 2022, pp. 425–443. MR 4458122
- Jacques Lewin, *Subrings of finite index in finitely generated rings*, J. Algebra **5** (1967), 84–88. MR 200297
- Andrea Lucchini and Attila Maróti, *Rings as the unions of proper subrings*, `http://arxiv.org/abs/1001.3984v1`, 2010.

References III

B. H. Neumann, *Groups covered by permutable subsets*, J. London Math. Soc. **29** (1954), 236–248. MR 62122

G. Peruginelli and N. J. Werner, *Maximal subrings and covering numbers of finite semisimple rings*, Comm. Algebra **46** (2018), no. 11, 4724–4738. MR 3864260

Gaetano Scorza, *I gruppi che possone pensarsi come somma di tre lori sottogruppi*, Boll. Un. Mat. Ital. **5** (1926), 216–218.

Eric Swartz and Nicholas J. Werner, *Covering numbers of commutative rings*, J. Pure Appl. Algebra **225** (2021), no. 8, 106622, 17. MR 4177971

______, *A new infinite family of σ-elementary rings*, Comm. Algebra **52** (2024), no. 1, 172–188. MR 4685762

References IV

M. J. Tomkinson, *Groups as the union of proper subgroups*, Math. Scand. **81** (1997), no. 2, 191–198. MR 1613772

Nicholas J. Werner, *Covering numbers of finite rings*, Amer. Math. Monthly **122** (2015), no. 6, 552–566. MR 3361734