The autogroup of the finitary power monoid of the integers under addition

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[1] Based on joint work with Kerou Wen (first-year graduate student at HebNU).

Outline

Power semigroups and power monoids

Below, unless otherwise stated, all sgrps are written multiplicatively

The large power sgrp of a sgrp S is the sgrp $\mathcal{P}(S)$ obtained by endowing the *non-empty* subsets of S with the (provably associative) operation

$$(X, Y) \mapsto XY := \{xy \colon x \in X,\, y \in Y\}.$$

If M is a monoid with identity 1_M, then $\mathcal{P}(M)$ is itself a monoid with identity $\{1_M\}$ and it is therefore called the large power monoid of M.

Each of the following is a submonoid of $\mathcal{P}(M)$:

- $\mathcal{P}_{\text{fin}}(M) := \{X \in \mathcal{P}(M) \colon |X| < \infty\}$, the finitary power monoid of M (the construction also makes sense for arbitrary sgrps $\longrightarrow$ finitary power sgrps).
- $\mathcal{P}_{\text{fin},1}(M) := \{X \in P_{\text{fin}}(M) : 1_M \in X\}$, the reduced finitary power monoid of M.

Depending on the context, these structures will be generically referred to as power semigroups or power monoids (shortly, PMs).

We will use $\mathcal{P}_{\text{fin},0}(M)$ instead of $\mathcal{P}_{\text{fin},1}(M)$ when M is written additively.

Origins

Power sgrps made their first *explicit* appearance in a 1953 paper by Dubreil[(2)]. They are an instance of the more abstract notion of power algebra and were studied quite intensively in the 1980s and 1990s.

A turning point in their history was marked by a 1967 paper by Takayuki Tamura & John Shafer[(3)], who were especially interested in the following:

Problem 1 (or "the Tamura–Shafer problem").

Given a class $\mathcal{O}$ of sgrps, prove/disprove that $\mathcal{P}(H) \cong \mathcal{P}(K)$, for arbitrary $H, K \in \mathcal{O}$, iff $H \simeq K$. (Here and later, $\cong$ means "is sgrp-isomorphic to".)

The crux of the problem lies in the "only if" direction, since every (sgrp) isomorphism $f \colon H \to K$ lifts to a global isomorphism $H \to K$ (i.e., to an isomorphism $\mathcal{P}(H) \to \mathcal{P}(K)$) via the map

$$X \mapsto f[X] := \{f(x) \colon x \in X\}.$$

This map is called the augmentation of f.

(2) Their definition is however *implicit* to the early work on additive number theory.
(3) Tamura & Shafer, Math. Japon. **12** (1967), 25–32.

Chasing the automorphisms

The Tamura–Shafer problem motivates the study of the automorphisms of power sgrps, leading to the following problem (among many others):

Problem 2.

Given a monoid M, "determine" the (sgrp) automorphisms of $\mathcal{P}_{\mathrm{fin},1}(M)$.

For each $f \in \mathrm{Aut}(M)$, the fnc

$$\mathcal{P}_{\mathrm{fin},1}(M) \to \mathcal{P}_{\mathrm{fin},1}(M) \colon X \mapsto f[X]$$

is a well-defined automorphism of $\mathcal{P}_{\mathrm{fin},1}(M)$, called the reduced finitary augmentation of f. An automorphism of $\mathcal{P}_{\mathrm{fin},1}(M)$ is inner if it is the reduced finitary augmentation of an automorphism of M. So, we have a map

$$\Phi \colon \mathrm{Aut}(M) \to \mathrm{Aut}(\mathcal{P}_{\mathrm{fin},1}(M))$$

sending an automorphism of M to its reduced finitary augmentation. In fact, Φ is an *injective* (*group*) *homomorphism* from $\mathrm{Aut}(M)$ to $\mathrm{Aut}(\mathcal{P}_{\mathrm{fin},1}(M))$.

One may ask whether Φ is also *surjective* and hence an isomorphism.

Progress

Trivially, the answer is already no when $M = (\mathbb{N}, +)$, as the reversion map $\mathrm{rev}\colon X \mapsto \max(X) - X$ is an automorphism of $\mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$. Less trivially:

Theorem 3.2 in [T. & Yan, JCTA 2025]

The reversion map is the only non-trivial automorphism of $\mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$.

The result prompted the following:

Conjecture (T. & Yan · JCTA 2025, Sect. 4)

If H is a numerical monoid $\neq \mathbb{N}$, then $\mathrm{Aut}(\mathcal{P}_{\mathrm{fin},0}(H))$ is trivial.

Rago and Yan have recently announced a proof of the conjecture [...].

In the meanwhile, Wen and T. considered a variant of Problem 2:

Problem 3.

Given a sgrp S, "determine" the autogroup of $\mathcal{P}_{\mathrm{fin}}(S)$.

We'll discuss a solution to Problem 3 when $S = \mathbb{Z}$ (additive group of integers).

Outline

1. Power semigroups

2. $\mathrm{Aut}(\mathcal{P}_{\mathrm{fin}}(\mathbb{Z})) \cong \mathbb{Z}_2 \times \mathrm{Dih}_\infty$

3. Prospects for future research

4. References

Totally ordered groups

A totally ordered [abelian] group is a pair $\mathcal{G} = (G, \preceq)$ consisting of a[n abelian] group G and a total order $\preceq$ on (the underlying set of) G such that $x \preceq y$ implies $uxv \preceq uyv$ for all $u, v \in G$. The set

$$\mathcal{G}_{\angle} := \{x \in G : 1_G \preceq x\}$$

is a submonoid of G, called the non-negative cone of $\mathcal{G}$.

For any non-empty finite set $X \subseteq G$, there exists a unique element $x_* \in X$ s.t. $x_* \preceq y$ for all $y \in X$. We refer to x_* as the $\preceq$-minimum of X.

The minimizer of $\mathcal{G}$ is the fnc $\mu_{\mathcal{G}} \colon \mathcal{P}_{\mathrm{fin}}(G) \to G$ sending a non-empty finite set $X \subseteq G$ to its $\preceq$-minimum. Note that, for all $X, Y \in \mathcal{P}_{\mathrm{fin}}(G)$,

$$\mu_{\mathcal{G}}(X)^{-1} X \in \mathcal{P}_{\mathrm{fin},1}(\mathcal{G}_{\angle}) \qquad \text{and} \qquad \mu_{\mathcal{G}}(XY) = \mu_{\mathcal{G}}(X)\mu_{\mathcal{G}}(Y).$$

In particular, $\mathbb{Z}$ is a totally ordered, abelian group under addition with respect to the standard ordering of the integers, and its non-negative cone is the set $\mathbb{N}$ of non-negative integers.

Annoying technicalities

For a *commutative* monoid M, define the relation of unit-associatedness by

$$\simeq_M = \{(x, y) \in M \times M : \exists\, u \in M^\times \text{ s.t. } y = xu\}.$$

This is a congruence on M, we denote the classes in the quotient by $[\cdot]_M$.

Proposition 2.1 in [T. & Wen, 202? · Proposition I.2.6 in Grillet, 2001]

Let H and K be commutative monoids. For any sgrp isomorphism $\varphi\colon H \to K$, there is an induced isomorphism $\widetilde{\varphi}\colon H/{\simeq_H} \to K/{\simeq_K}$ defined by:

$$\widetilde{\varphi}([x]_H) = [\varphi(x)]_K, \qquad \text{for every } x \in H.$$

Proposition 2.3 in [T. & Wen, 202?]

Let $\mu_{\mathcal{G}}$ be the minimizer of a totally ordered abelian group $\mathcal{G} = (G, \preceq)$, written additively. Then the binary relation $\psi\colon \mathcal{P}_{\text{fin}}(G)/{\simeq_G} \to \mathcal{P}_{\text{fin},1}(\mathcal{G}_\angle)$ defined by:

$$[X]_{\mathcal{P}_{\text{fin}}(H)} \mapsto Y - \mu_{\mathcal{G}}(Y), \qquad \text{for all } Y \simeq_{\mathcal{P}_{\text{fin}}(G)} X,$$

is in fact a (*monoid*) isomorphism.

Splitting hairs

From the previous two propositions, we obtain:

Proposition 2.4 in [T. and Wen, 202?]

Let $\mathcal{H} = (H, \le)$ and $\mathcal{K} = (K, \preceq)$ be totally ordered abelian groups (written additively), $\mu_{\mathcal{H}}$ be the minimizer of $\mathcal{H}$, and φ be an isomorphism from $\mathcal{P}_{\mathrm{fin}}(H)$ to $\mathcal{P}_{\mathrm{fin}}(K)$. There exist an isomorphism f from $\mathcal{P}_{\mathrm{fin},1}(\mathcal{H}_\angle)$ to $\mathcal{P}_{\mathrm{fin},1}(\mathcal{K}_\angle)$ and a homomorphism α from $\mathcal{P}_{\mathrm{fin}}(H)$ to K such that

$$\varphi(X) = f(X - \mu_{\mathcal{H}}(X)) + \alpha(X), \qquad \text{for all } X \in \mathcal{P}_{\mathrm{fin}}(H).$$

In the notation of Proposition 2.3 from the previous slide:

$$\begin{tikzcd}
\mathcal{P}_{\mathrm{fin}}(H) \arrow[rr, "\varphi"] \arrow[drr, phantom, ""] & & \mathcal{P}_{\mathrm{fin}}(K) \\
\mathcal{P}_{\mathrm{fin}}(H)/{\simeq_{\mathcal{P}_{\mathrm{fin}}(H)}} \arrow[rr, "\widetilde{\varphi}"'] \arrow[d, "\psi_H"'] & & \mathcal{P}_{\mathrm{fin}}(K)/{\simeq_{\mathcal{P}_{\mathrm{fin}}(K)}} \arrow[d, "\psi_K"] \\
\mathcal{P}_{\mathrm{fin},1}(\mathcal{H}_\angle) \arrow[rr, "f"] & & \mathcal{P}_{\mathrm{fin},1}(\mathcal{K}_\angle)
\end{tikzcd}$$

Main theorem

Assume now that the monoids H and K in the last proposition are both equal to $\mathbb{Z}$ (the additive group of integers). We have that

- the non-negative cone of $\mathbb{Z}$ under the usual ordering is $\mathbb{N}$;
- the homomorphisms $\mathcal{P}_{\mathrm{fin}}(\mathbb{Z}) \to \mathbb{Z}$ are precisely the maps of the form

$$X \mapsto \alpha \min X + \beta \max X, \qquad \text{with } \alpha, \beta \in \mathbb{Z};$$

- the only non-trivial automorphism of $\mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$ is the map rev (Slide 6).

Stitching the pieces together, we obtain the following:

Theorem 3.4 in [T. & Wen, 202?]

The automorphisms of $\mathcal{P}_{\mathrm{fin}}(\mathbb{Z})$ are precisely the endofunctions of $\mathcal{P}_{\mathrm{fin}}(\mathbb{Z})$ of the form $\pm f_\alpha$ and $\pm g_\alpha$, where $\alpha \in \mathbb{Z}$ is arbitrary and, for all $X \in \mathcal{P}_{\mathrm{fin}}(\mathbb{Z})$,

$$f_\alpha(X) := X + \alpha \min X - \alpha \max X$$

and

$$g_\alpha(X) := X + (\alpha - 1) \min X - (\alpha + 1) \max X.$$

Proof sketch

- Wlog, every $\varphi \in \mathrm{Aut}(\mathcal{P}_{\mathrm{fin}}(\mathbb{Z}))$ satisfies, by Proposition 2.4 and the comments before Theorem 3.4, the following condition:

$$\varphi(X) = X + a\min X + b\max X, \qquad \text{for all } X \in \mathcal{P}_{\mathrm{fin}}(\mathbb{Z}). \tag{1}$$

  Here, a and b are unknown integers to be determined by imposing that φ is bijective (and hence an automorphism).

- Imposing $\varphi(X) = \{1\}$ and taking the min and the max of both sides of Eq. (1) results in a system of two linear Diophantine equations in the variables $y := \min X$ and $z := \min Y$:

$$\begin{cases} (a+1)y + bz = 1, \\ ay + (b+1)z = 1. \end{cases}$$

- It follows that $a + b = 0$ or $a + b = -2$, and hence (in the notation of the statement) there exists $\alpha \in \mathbb{Z}$ s.t. $\varphi = f_\alpha$ or $\varphi = g_\alpha$.

- On the other hand, recognizing that f_0 is the identity fnc on $\mathcal{P}_{\mathrm{fin}}(\mathbb{Z})$ yields

$$f_{\pm\alpha} \circ f_{\mp\alpha} = g_\alpha \circ g_\alpha = f_0,$$

  and it's easy to check that f_α and g_α are also endomorphisms. ■

A characterization

The infinite dihedral group Dih_∞ is defined as:

$$\mathrm{Dih}_\infty := \mathbb{Z}_2 \ltimes_\alpha \mathbb{Z} \cong \mathrm{Grp}\langle x, y \mid x^2 = y^2 = 1 \rangle,$$

where $\mathbb{Z}_2$ is the group of units $\{\pm 1\}$ of the integers, and α is the unique group homomorphism $\mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$ sending -1 to the only non-trivial automorphism of $\mathbb{Z}$ (i.e., the inversion map $x \mapsto -x$).

Theorem 3.5 in [T. & Wen, 202?]

$\mathrm{Aut}(\mathcal{P}_{\mathrm{fin}}(\mathbb{Z})) \cong \mathbb{Z}_2 \times \mathrm{Dih}_\infty$.

Proof sketch.

Given $\alpha \in \mathbb{Z}$, let f_α and g_α be defined as in Theorem 3.4. Accordingly, set

$$E := \{f_0, g_0\}, \quad F := \{f_\alpha : \alpha \in \mathbb{Z}\}, \quad G := \{g_\alpha : \alpha \in \mathbb{Z}\}, \quad \text{and } H := F \cup G.$$

We have $E \trianglelefteq H \le \mathrm{Aut}(\mathcal{P}_{\mathrm{fin}}(\mathbb{Z}))$, $F \le H = EF$, and $E \cap F = \{f_0\}$, which, by definition, implies that $H = E \ltimes F$. On the other hand, Example (I) on p. 51 of [Robinson, *A Course in the Theory of Groups*, 1996] implies $E \ltimes F \cong \mathrm{Dih}_\infty$.

Since the map $(\pm 1, \phi) \mapsto \pm\phi$ provides an isomorphism $\mathbb{Z}_2 \times H \to \mathrm{Aut}(\mathcal{P}_{\mathrm{fin}}(\mathbb{Z}))$, it follows by the elementary properties of direct products that $\mathrm{Aut}(\mathcal{P}_{\mathrm{fin}}(\mathbb{Z})) \cong \mathbb{Z}_2 \times H \cong \mathbb{Z}_2 \times \mathrm{Dih}_\infty$. ∎

Outline

Prospects for future work

Returning to Problem 2, let $(G, \preceq)$ be a totally ordered group.

Problem 1.

Is every automorphism φ of the reduced finitary power monoid $\mathcal{P}_{\mathrm{fin},1}(G)$ ***inner****?*

That is, does there exist an automorphism f of G such that

$$\varphi(X) = f[X], \qquad \text{for all } X \in \mathcal{P}_{\mathrm{fin},1}(G)?$$

In particular, does this hold when G is abelian?

By work of Rago and Yan, the question has an affirmative answer for the case when G is a subgroup of the additive group of $\mathbb{Q}$.

Problem 2.

Given a group G, does there exists a monoid M s.t. $\mathrm{Aut}(\mathcal{P}_{\mathrm{fin},1}(M)) \cong G$? If not, characterize the groups for which this is true.

Rago has recently shown that, for every finite abelian group G, the autogroup of $\mathcal{P}_{\mathrm{fin},1}(G)$ is isomorphic to $\mathrm{Aut}(G)$, except when G is the Klein four-group, in which case $\mathrm{Aut}(\mathcal{P}_{\mathrm{fin},1}(G)) \cong S_3 \times S_3$.

Outline

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