On the Ideal Theory of HNP Rings

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Commutative Dedekind Domains

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- coordinate rings $k[C]$ of non-singular affine algebraic curves C over an a.c. field k

Ideal Factorization Theory

Theorem

Any nonzero ideal $I \lhd D$ factors as

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for some maximal ideals $P_i \lhd D$ in a unique way.

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$$\partial \colon \text{ideals in } D \longrightarrow \bigoplus_{M \text{ simple}} \mathbb{Z}_{\geq 0} \cdot M = \mathrm{Div}(D)$$

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$$\mathbb{H}_{\mathbb{Z}} \subseteq \mathbb{H}_{\mathbb{Z}} + \frac{1}{2}(1 + i + j + k)\,\mathbb{Z}$$

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Theorem (Rump-Yang 2016, 2025)

There is a binary operation $\circ$ on $\mathrm{Div}(R)$ such that

$$\partial\colon \text{two-sided ideals in } R \longrightarrow (\mathrm{Div}(R), \circ)$$

is a monoid embedding.

One-Sided Ideal Theory

We shouldn't just multiply ${}_R I$ and ${}_R J$!

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Category of one-sided ideals:

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\end{tikzcd}
$$

Theorem (Smertnig-V. 2025)

There is at most **one** operation $\circ$ that makes ∂ a functor such that $D \circ \_ - D\colon \mathrm{Div}(R) \to \mathrm{Div}(S)$ is additive.

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For ideals I and J, we have: $\partial IJ = \partial I + I \otimes \partial J$.

References

- L. S. Levy and J. C. Robson, *Hereditary Noetherian prime rings and idealizers* No. **174**, Amer. Math. Soc. 2011
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