The Covering Numbers of Rings II

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Rings, Subrings, and Covers

For this talk:

- A ring R is an associative ring with unity.
- A subring S of R need not have a multiplicative identity.
  Subring $=$ group under $+$, closed under mult.

Let R be a ring.

- A cover of R is a collection $\mathcal{C}$ of proper subrings of R whose union is all of R:

$$R = \bigcup_{S \in \mathcal{C}} S$$

- R is coverable if and only if a cover exists.
- The covering number of R is the minimum number of subrings necessary to cover R.
- $\sigma(R) =$ covering number of R. If R is not coverable, then we set $\sigma(R) = \infty$

σ-elementary Rings

Recall: $\sigma(R) \leqslant \sigma(R/I)$ for any two-sided ideal I

Definition

R is σ-elementary if $\sigma(R) < \sigma(R/I)$ for every non-zero two-sided ideal of R.

- A σ-elementary ring must be coverable, since $\sigma(R) < \sigma(\{0\}) = \infty$.
- If R is coverable but **not** σ-elementary, then $\sigma(R) = \sigma(R/I)$ for some I.
- If R admits a finite cover, then $\sigma(R) = \sigma(R')$ for some σ-elementary ring R' that is a residue ring of R.
- If $n \in \mathbb{N}$ occurs as a covering number of a ring, then n is the covering number of some σ-elementary ring.

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Goal: classify all σ-elementary rings

The Big Theorem: Four Families of σ-elementary Rings

Theorem (Swartz, W. (2024))

Let R be a σ-elementary ring.
Then, R is a finite ring of characteristic p, and one of the following holds.

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   For some prime power q and integer $n \geqslant 2$, $R \cong M_n(q)$

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4. If R is noncommutative and not semisimple, then R is a ring of AGL-type.

Summary of Formulas

Let R be a σ-elementary ring.

1. If R is commutative and semisimple, then $R \cong \bigoplus_{i=1}^{\tau(q)} \mathbb{F}_q$, $q = p^d$, and
$$\sigma(R) = \tau(q)\nu(q) + d\binom{\tau(q)}{2}$$

2. If R is commutative but not semisimple, then $R \cong \mathbb{F}_q(+)\mathbb{F}_q^2$, and $\sigma(R) = q + 1$

3. If R is noncommutative and semisimple, then $R \cong M_n(q)$, a is the smallest prime divisor of n, and
$$\sigma(R) = \frac{1}{a}\prod_{k=1, a\nmid k}^{n-1}(q^n - q^k) + \sum_{k=1, a\nmid k}^{\lfloor n/2 \rfloor}\binom{n}{k}_q$$

4. If R is noncommutative and not semisimple, then (barring small exceptions), $R \cong A(n, q_1, q_2)$, $n \geqslant 3$, $q = q_1^d$, $d < n$, and
$$\sigma(R) = q^n + \binom{n}{d}_{q_1} + \omega(d)$$

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All possible covering numbers of rings come from these formulas!!

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Bounds: $\frac{q^2}{d^2} \leqslant \sigma(R) \leqslant \frac{q^2}{2d}$

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Bounds: $q^{n(n-(n/a)-1)} \leqslant \sigma(R) \leqslant q^{n(n-(n/a))}$

4. If R is noncommutative and not semisimple, then (barring small exceptions), $R \cong A(n, q_1, q_2)$, $n \geqslant 3$, $q = q_1^d$, $d < n$, and
$$\sigma(R) = q^n + \binom{n}{d}_{q_1} + \omega(d)$$
Bounds: $q^n \leqslant \sigma(R) \leqslant q^n + q^{n-d+1} + d$

All possible covering numbers of rings come from these formulas!!

Reductions 1 and 2: Finite Rings of Order p^n

Theorem (B. H. Neumann, J. Lewin)

Let R be a coverable ring such that $\sigma(R)$ is finite. Then, there exists a two-sided ideal I of R such that R/I is finite and $\sigma(R) = \sigma(R/I)$.

Chinese Remainder Theorem

Let R be a finite ring. Then, R is isomorphic to a direct product of rings of prime power order:

$$R \cong R_1 \times R_2 \times \cdots \times R_m, \quad \text{where } |R_i| = p_i^{d_i} \text{ for distinct primes } p_1, \ldots, p_m.$$

Also, if S is a subring of R, then $S \cong S_1 \times S_2 \times \cdots \times S_m$, where each S_i is a subring of R_i.

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Corollary

Let R be as above. If R is coverable, then $\sigma(R) = \min_{1 \leqslant i \leqslant m} \sigma(R_i)$.

Reduction 3: Rings of Characteristic p

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Proof sketch:

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Proof sketch:

- Show that $pR \subseteq M$ for **every** maximal subring M of R.
- Since pR is contained in every maximal subring, any minimal cover of R can be pushed forward onto R/pR.

  So, $\sigma(R/pR) \leqslant \sigma(R)$.
- But, $\sigma(R) \leqslant \sigma(R/pR)$. Thus, $\sigma(R/pR) = \sigma(R)$.

Reduction 4: Rings with $\mathrm{JRad}(R)^2 = 0$

Wedderburn-Malcev Theorem

Let R be a finite ring characteristic p.
Let J be the Jacobson radical of R (J = intersection of max. ideals).
There exists a semisimple (direct sum of mat. rings) $\mathbb{F}_p$-subalgebra S of R such that:

- $R = S \oplus J$
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Corollary

If $\sigma(R)$ is finite, then to find $\sigma(R)$ it suffices to consider residue rings of R that are finite; have characteristic p; and have $J^2 = 0$.

The “Too Many Conjugates” Principle

Proposition (Swartz, W. (2021))

Let R be a σ-elementary ring, and let $\mathcal{C}$ be a minimal cover of R.

Let T be a maximal subring of R that has an ideal complement in R.

That is $R = T \oplus I$ for some two-sided ideal I of R.
Then

$$\# \text{ conjugates of } T \text{ in R} \leqslant \sigma(R) < \sigma(T)$$

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- If $T \notin \mathcal{C}$, then contract each subring in $\mathcal{C}$ to T to get a cover of T. So,

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Contradiction!

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How to use this: If T has "too many" conjugates, then R is **not** σ-elementary!

Peirce Decomposition

Assumptions so far:

- R: finite, characteristic p
- J: Jacobson radical of R, $J^2 = 0$
- $R = S \oplus J$, where $S \cong R/J$ is semisimple
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Then, for some $N \geqslant 1$,

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$$J = \bigoplus_{1 \leqslant i,j \leqslant n} e_i J e_j$$

Each $e_i J e_j$ is an (S_i, S_j)-bimodule.

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  Each $e_i J e_j$ is an (S_i, S_j)-bimodule.
- Because $J^2 = 0$, each $e_i J e_j$ is a two-sided ideal of R!!

Restrictions on σ-elementary Rings

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Consequences of Peirce Decomposition of J:

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- Assume $\mathcal{I}$ is a maximal subideal of J. Let $T = S \oplus \mathcal{I}$.
  Then, T is a maximal subring of R, and $R = T \oplus \widehat{\mathcal{I}}$, and $T \cong R/\widehat{\mathcal{I}}$.

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$$\# \text{ conjugates of } T \text{ in R} \leqslant \sigma(R) < \sigma(T)$$

- Roughly speaking, we end up with lots of conjugates whenever:
  - S_i is noncommutative and $e_i J e_i \neq 0$; or
  - Both S_i and S_j are noncommutative and $e_i J e_j \neq 0$; or
  - S_i is noncommutative, S_j is a field, $e_i J e_j$ is not a simple (S_i, S_j)-bimodule

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  R cannot be σ-elementary in these cases

After the Dust Settles

Assume R is σ-elementary.

- There is at most one pair (i, j) with $e_i J e_j \neq 0$.

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    WLOG, $S_i = M_n(q_i)$ and $S_j = \mathbb{F}_{q_j}$.
    Then, $e_i J e_j$ must be a simple (S_i, S_j)-bimodule:

$$e_i J e_j \cong M_{n\times 1}(q), \quad q = \text{order of compositum of } \mathbb{F}_{q_i} \text{ and } \mathbb{F}_{q_j}$$

We get $R \cong \left(\begin{array}{c|c} M_n(q_i) & M_{n\times 1}(q) \\ \hline 0 & \mathbb{F}_{q_j} \end{array}\right)$

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The Dorroh extension or unitization of R is the ring

$$R^1 := \mathbb{Z}_n \times R$$

with operations

$$(n_1, r_1) + (n_2, r_2) = (n_1 + n_2,\ r_1 + r_2)$$
$$(n_1, r_1)(n_2, r_2) = (n_1 n_2,\ n_1 r_2 + n_2 r_1 + r_1 r_2)$$

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Properties:

- R^1 is a ring with identity element $(1, 0)$
- R embeds into R^1 via $r \mapsto (0, r)$

Unital Covers

Definition

Let R be a ring with unity. Assume R can be covered by proper subrings containing 1_R.

Let $\sigma_u(R)$ be the size of a minimal cover of R by proper subrings containing 1_R.

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2. Let R be a rng that has a finite cover.
   Let R^1 be the Dorroh extension of R.
   Then, $\sigma(R) = \sigma_u(R^1)$.

Summary of Formulas

Let R be a σ-elementary ring with unity.

1. If R is commutative and semisimple, then $R \cong \bigoplus_{i=1}^{\tau(q)} \mathbb{F}_q$, $q = p^d$, and

$$\sigma(R) = \tau(q)\nu(q) + d\binom{\tau(q)}{2}$$

Bounds: $\frac{q^2}{d^2} \leqslant \sigma(R) \leqslant \frac{q^2}{2d}$

2. If R is commutative but not semisimple, then $R \cong \mathbb{F}_q(+)\mathbb{F}_q^2$, and $\sigma(R) = q + 1$

3. If R is noncommutative and semisimple, then $R \cong M_n(q)$, a is the smallest prime divisor of n, and

$$\sigma(R) = \frac{1}{a}\prod_{k=1, a\nmid k}^{n-1}(q^n - q^k) + \sum_{k=1, a\nmid k}^{\lfloor n/2 \rfloor} \binom{n}{k}_q$$

Bounds: $q^{n(n-(n/a)-1)} \leqslant \sigma(R) \leqslant q^{n(n-(n/a))}$

4. If R is noncommutative and not semisimple, then (barring small exceptions), $R \cong A(n, q_1, q_2)$, $n \geqslant 3$, $q = q_1^d$, $d < n$, and

$$\sigma(R) = q^n + \binom{n}{d}_{q_1} + \omega(d)$$

Bounds: $q^n \leqslant \sigma(R) \leqslant q^n + q^{n-d+1} + d$

All possible covering numbers of rings come from these formulas!!

Almost All Positive Integers are Not Covering Numbers

Theorem (Swartz, W. (2024))

Let $N \in \mathbb{N}$, $N \geqslant 2$.

Form	Max # integers at most N
$\tau(q)\nu(q) + d\binom{\tau(q)}{2}$	
$q + 1$	
$\frac{1}{a}\prod_{k=1, a\nmid k}^{n-1}(q^n - q^k) + \sum_{k=1, a\nmid k}^{\lfloor n/2 \rfloor}\binom{n}{k}_q$	
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Corollary

Define $\mathscr{E}(N) := \{m \mid m \leqslant N,\ \sigma(R) = m \text{ for some ring } R\}$.

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Define $\mathscr{E}(N) := \{m \mid m \leqslant N,\ \sigma(R) = m \text{ for some ring } R\}$.

1. $|\mathscr{E}(N)| \leqslant \frac{144N}{\log_2 N}$
2. $\lim\limits_{N\to\infty} \frac{|\mathscr{E}(N)|}{N} = 0$
3. Almost all positive integers are not the covering number of a ring

THANK YOU!

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