On the structure of length sets with maximal elasticity

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Outline

1. Factorization Theory and Krull monoids
2. Monoids of zero-sum sequences and the Davenport constant
3. Transfer homomorphisms and transfer Krull monoids
4. Main results on length sets with maximal elasticity

- Let $(H, \cdot)$ be a monoid, that is, a commutative, cancellative semigroup with the identity element.

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- For a finite nonempty set $L \subset \mathbb{N}$, we denote by $\rho(L) = \max L / \min L$ the elasticity of L, and by $\rho(H)$, defined as the supremum of $\rho\big(\mathsf{L}(a)\big)$ over all a of H, the elasticity of H.

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A monoid H is a *Krull monoid* if one of the following equivalent conditions is satisfied.

(a) There is a divisor homomorphism $\varphi\colon H \to D$, where D is a free abelian monoid such that for every $\alpha \in D$ there is a finite nonempty set $A \subset H$ such that $\alpha = \gcd \varphi(A)$.

(b) There is a divisor homomorphism from H into a free abelian monoid.

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Theorem

Let D be a domain. Then $D^\bullet$ is a Krull monoid if and only if D is a Krull domain. Integrally closed Notherian domain is a Krull domain.

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Let H and B be atomic monoids. Then a monoid homomorphism $\theta : H \longrightarrow B$ is called a *transfer homomorphism* if the following properties are satisfied.

1) $B = \theta(H)B^\times$ and $\theta^{-1}(B^\times) = H^\times$.

2) If $u \in H, b, c \in B$ and $\theta(u) = bc$, then there exist $v, w \in H$ such that $u = vw, \theta(v) \simeq b$ and $\theta(w) \simeq c$.

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- Transfer homomorphisms preserve the arithmetic structure, particularly, they preserve the sets of lengths and elasticity.

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$\mathcal{B}(G_0)$ is a Krull monoid

Since the inclusion $\mathcal{B}(G_0) \hookrightarrow \mathcal{F}(G_0)$ is a divisor homomorphism, Condition (b) of the Definition implies that $\mathcal{B}(G_0)$ is a Krull monoid.

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- Maximal order in central simple algebras over number fields.

The Davenport constant of G

- Let G be a finite abelian group, say $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $r \in \mathbb{N}$ and $1 < n_1 | \ldots | n_r$, and set $\mathrm{D}^*(G) = 1 + \sum_{i=1}^{r}(n_i - 1)$.

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- So far, the groups $C_2^4 \oplus C_{2k}$ with $k \ge 70$ is odd, is only one series of groups with $\mathsf{D}^*(G) < \mathsf{D}(G)$, for which the precise value of the Davenport constant is known.

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- So far, the groups $C_2^4 \oplus C_{2k}$ with $k \ge 70$ is odd, is only one series of groups with $\mathsf{D}^*(G) < \mathsf{D}(G)$, for which the precise value of the Davenport constant is known.
- Let H be a Krull monoid with finite class group G and suppose that each class contains a prime divisor. Then it is known that $\rho(H) = \mathsf{D}(G)/2$.

Main results

Theorem (2025)

Let H be a transfer Krull monoid over a finite abelian group G with $|G| \geq 3$. Suppose that G has the following Property **P**.

> **P.** There are $g_1, g_2 \in G$ and minimal zero-sum sequences U_1, U_2 over G such that
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> $$|U_1| = |U_2| = D(G), \; g_1 g_2 \mid U_1, \quad \text{and} \quad (g_1 + g_2) \mid U_2 \, .$$

Then there exists some $a^* \in H$ such that for all elements $a \in H$ with $\rho(\mathsf{L}(a)) = \rho(H)$, the length set $\mathsf{L}(a^* a)$ is an interval with elasticity $\rho(H)$.

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In general, length sets $\mathsf{L}(a)$ with elasticity $\rho(H)$ need not be intervals (in other words, the above statement does not hold for $a^* = 1$).

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is satisfied in each of the following cases.

(a) G is a cyclic group of odd order.

(b) G is not cyclic and $\mathsf{D}(G) = \mathsf{D}^*(G)$.

(c) G has odd order and there is some $U \in \mathcal{A}(\mathcal{B}(G))$ of length $|U| = \mathsf{D}(G)$ that is not squarefree.

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Note that there is known no non-cyclic group that does not satisfy Property **P**.

On Property $\mathbf{P}^*$

- Define Property $\mathbf{P}^*$ as follows: For every nonzero element $g \in G$, there is some $A_g \in \mathcal{A}(\mathcal{B}(G))$ with $|A_g| = \mathsf{D}(G)$ and $g \in \operatorname{supp}(A_g)$.

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Example

Suppose that G is an elementary p group of rank r and let $g = e_1 \in G \setminus \{0\}$. Then e_1 can be extended to a basis, say $(e_1, e_2, \ldots, e_r)$. Then

$$A_g = (e_1 + \ldots + e_r) \prod_{i=1}^{r} e_i^{p-1}$$

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Let H be a transfer Krull monoid over a finite abelian group G satisfying Property $\mathbf{P}^*$. Then there exists some $a^* \in H$ such that all elements $a \in H$ with $\rho(\mathsf{L}(a)) = \rho(H)$, length sets $\mathsf{L}(a^* a)$ are intervals with elasticity $\rho(H)$.

On cyclic group of even order

Theorem (2025)

Let H be a transfer Krull monoid over a cyclic group G of even order $|G| \geq 4$.

1. There is no element $a \in H$ with maximal elasticity such that $\max \mathsf{L}(a) - 1 \in \mathsf{L}(a)$.
2. If $|G| + 1 \notin \mathbb{P}$, then there exist $a^* \in H$ and $M \in \mathbb{N}_0$ such that for all $a \in H$ with $\rho(\mathsf{L}(a)) = \rho(H)$, $\mathsf{L}(a^* a) \cap [\min \mathsf{L}(a^* a), \max \mathsf{L}(a^* a) - M]$ is an interval and $\rho(\mathsf{L}(a^* a)) = \rho(H)$.

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- Particularly, this theorem implies that there is no element $a \in H$ with maximal elasticity such that $\mathsf{L}(a)$ is an interval.
- This case is the only known exceptional case.

Density Result (work in progress)

Corollary (2025)

If G has Property **P**, then

$$\lim_{n\to\infty} \frac{\left|\left\{A \in \mathcal{B}(G) : \ |A| \leq n, \ \ \mathsf{L}(A) \text{ is an interval with } \rho(\mathsf{L}(A)) = \mathsf{D}(G)/2\right\}\right|}{\left|\left\{A \in \mathcal{B}(G) : |A| \leq n, \ \ \rho(\mathsf{L}(A)) = \mathsf{D}(G)/2\right\}\right|} = 1$$

Thank you for your attention!