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On invertible-radical factorization in commutative rings

Youssef ZAHIR

University Mohamed V in Rabat
Faculty of Sciences of Rabat
Department of Mathematics
Morocco.

Joint work with M. T. Ahmed and N. Mahdou

- **General context:** All rings are commutative with a nonzero unit and all modules are unitary.
- T. Dumitrescu and M. T. Ahmed introduced and studied the notion of an ISP-domain, that is, integral domain whose ideals can be factored as an invertible ideal times a nonempty product of proper radical ideals (this terminology comes from ”invertible semiprime ideal”)
- Our aim is to extend the ISP-domain concept to rings with zero-divisors in two different ways.

ISP-rings

Definition

A is said to be an ISP-ring if every proper regular ideal of A can be factored as an invertible ideal times a nonempty product of proper radical ideals.

Proposition

Let B be a finite direct product of some family of rings $(A_i)_{i=1,\dots,n}$. Then B is an ISP-ring if and only if each A_i is an ISP-ring.

ISP-rings

The trivial ring extension of a ring A by an A-module E (also called the idealization of E over A) is the ring $A \propto E$ whose underlying group is $A \times E$ with multiplication given by

$$(a, e)(b, f) = (ab, af + be).$$

For more details on trivial ring extensions, we refer the reader to Glaz's and Huckaba's respective books.

ISP-rings

Proposition

Let $f : A \longrightarrow B$ be a surjective ring homomorphism such that $Ker(f)$ is a prime ideal of A. Assume that every regular ideal of B can be lifted via f to a regular ideal of A. If A is an ISP-ring, then so is B.

Remark

Note that the condition "$Ker(f)$ is a prime ideal" is necessary. Indeed, let (A, M) be a non-Prüfer local domain, $K = qf(A)$ and $R = A \propto K$. By Corollary 2.8 and [2, Theorem 5], R is not an ISP-ring. Now, let E be a nonzero vector space over $\frac{R}{M \propto K}$ and $T = R \propto E$. Clearly, T is an ISP-ring, but $\frac{T}{0 \propto E}$ ($\simeq R$) is not an ISP-ring.

ISP-rings

Let P be a prime ideal of a ring A. Denote by $A_{(P)} = \{a/b \in T(A) | a \in A, b \in A \setminus P$ and b is regular$\}$ the regular localization of A at P. Here $T(A)$ denotes the total quotient ring of A.

Proposition

Let A be an ISP-ring and S a multiplicatively closed set which consists only of regular elements. Then A_S is an ISP-ring (in particular, $A_{(P)}$ is an ISP-ring, where P is a prime ideal of A).

Theorem

Let A be a ring and E an A-module such that $E = sE$ for every $s \in S = A \setminus (Z(A) \cup Z(E))$. Then $A \propto E$ is an ISP-ring if and only if every proper ideal of A not disjoint to S can be factored as an invertible ideal times a nonempty product of proper radical ideals.

Corollary

Let A be an integral domain with quotient field K, E a K-vector space, and $R = A \propto E$. Then R is an ISP-ring if and only if A is an ISP-domain.

Example

Let $R = \mathbb{Z} \propto \mathbb{Q}$ be a trivial ring extension of $\mathbb{Z}$ by $\mathbb{Z}$-module $\mathbb{Q}$. By Corollary, R is an ISP-ring.

The amalgamated duplication of a ring A along an ideal I, introduced and studied by D'Anna and denoted by $A \bowtie I$, is the following subring of $A \times A$ (endowed with the usual componentwise operations):

$$A \bowtie I = \{(a, a+i) | a \in A \ \ and \ \ i \in I\}.$$

Note that if $I^2 = 0$, then this construction $A \bowtie I$ coincides with the trivial ring extension $A \propto I$.
Next, we present our main result about the transfer of the ISP-ring property to amalgamated duplication of a ring along an ideal.

Theorem

Let A be a ring and I an ideal of A. Then:

1. If $A \bowtie I$ is an ISP-ring then so is A.
2. Assume that $I = aI$ for each $a \in Reg(A)$. Then $A \bowtie I$ is an ISP-ring if and only if so is A.

To prove theorem, we need the followings lemmas.

Lemma

Let A be a ring and I an ideal of A. Then the following statements are equivalent:

1. Every regular ideal of $A \bowtie I$ has the form $H \bowtie I$, where H is a regular ideal of A.
2. $I = aI$ for every $a \in Reg(A)$.

Lemma

Let A be a ring and I, J two ideals of A. If $J \bowtie I$ is invertible, then so is J.

Lemma

Let A be a ring and I, J two ideals of A. If J is regular finitely generated and $I = aI$ for each $a \in Reg(A)$, then $J \bowtie I$ is a finitely generated ideal of $A \bowtie I$.

Example

Let $A = \mathbb{Z} \propto \mathbb{Q}$ be a trivial ring extension of $\mathbb{Z}$ by $\mathbb{Z}$-module $\mathbb{Q}$ and $I = 0 \propto \mathbb{Q}$ an ideal of A. Then $A \bowtie I$ is an ISP-ring. Indeed, by the previous Example, A is an ISP-ring. Since $I = (n, q)I$ for each $(n, q) \in Reg(A)$, therefore $A \bowtie I$ is an ISP ring by the above Theorem.

Remark

We cannot obtain nontrivial examples of ISP-rings $A \bowtie I$ starting with a total quotient ring A. Indeed, if A is a total quotient ring, then so is $A \bowtie I$ (and hence it is an ISP-ring). To see this, if $(a, b) \in A \bowtie I$ is a regular element, then a, b are regular elements in A (hence units) and $a - b \in I$. Let $aa' = 1$ and $bb' = 1$ with $a', b' \in A$. Then $a' - b' \in I$, so $(a', b') \in A \bowtie I$.

Strongly ISP-rings

Definition

We call A a strongly ISP-ring if every proper ideal of A can be factored as an invertible ideal times a nonempty product of proper radical ideals.

Clearly, strongly ISP-domains are exactly ISP-domains. ZPUI and von Neumann regular rings are trivial examples of strongly ISP-rings.

Strongly ISP-rings

Note that every strongly ISP-ring is an ISP-ring. The converse is not true in general, as the following example shows.

Example

Let (A, M) be a local ring which is not reduced and E a nonzero A-module such that $ME = 0$. Then $A \propto E$ is an ISP-ring which is not strongly ISP. Indeed, clearly $A \propto E$ is a total quotient ring and hence an ISP-ring. Now, assume that $0 \propto E = (J \propto E)(H_1 \propto E) \cdots (H_n \propto E)$ with $J \propto E$ an invertible ideal, $n \geq 1$ and all $H_i \propto E's$ are proper radical ideals. If $n = 1$ we get $JH_1 = 0$ and hence $H_1 = 0$ since J is an invertible ideal of A, a contradiction. If $n > 1$ we get $E = 0$, again a contradiction.

Strongly ISP-rings

Proposition

The following assertions hold:

1. If A is a strongly ISP-ring and P a prime ideal of A, then A/P is an ISP-domain.
2. If S is a multiplicatively closed set of a strongly ISP-ring A, then A_S is a strongly ISP-ring.
3. A finite direct product of some family of rings $(A_i)_{i=1,\ldots,n}$ is a strongly ISP-ring if and only if each A_i is a strongly ISP-ring.

Strongly ISP-rings

Recall that a ring A is called special primary if $Spec(A) = \{M\}$ and each proper ideal of A is a power of M. Note that zero-dimensional rings are total quotient, that is, they have no non-unit regular element.

Proposition

Let A be a zero-dimensional local strongly ISP-ring with maximal ideal M. Then A is special primary.

Strongly ISP-rings

Recall that an almost multiplication ring is a ring whose localizations at its prime ideals are discrete rank one valuation domains or special primary rings.

Theorem

Let A be a strongly ISP-ring such that every nonzero prime ideal of A is maximal. Then A is almost multiplication.

Recall that a ring A is ZPI if every proper ideal of A is a product of prime ideals.

Corollary

For a ring A the following assertions are equivalent.

1. A is a ZPI-ring.
2. A is a Noetherian SSP-ring.
3. A is a Noetherian strongly ISP-ring.

Strongly ISP-rings

Theorem

Let A be a ring and E an A-module

1. If $A \propto E$ is a strongly ISP-ring, then so is A.
2. If A is a von Neumann regular ring and E is a multiplication Amodule, then $A \propto E$ is a strongly ISP-ring.
3. If $A \propto E$ is a strongly ISP-ring and $E = sE$ for each $s \in S$, then E is a multiplication module, where $S = A \setminus (Z(A) \cup Z(E))$.

Strongly ISP-rings

We get the following result, where $Supp(E)$ denotes the support of an A-module E.

Theorem

Let $A \propto E$ be a strongly ISP-ring in which every prime ideal is maximal. Then for each maximal ideal $M \in Supp(E)$, A_M is a field and $E_M \simeq A_M$.

Strongly ISP-rings

Recall that an A-module E is simple if it has no proper nonzero submodule. Moreover, E is called divisible if for every regular element $a \in A$ and for every $e \in E$ there exists $e' \in E$ such that $e = ae'$. Equivalently, $E = aE$ for every regular element $a \in A$.

Proposition

Let A be an integral domain and E a divisible A-module. Then $A \propto E$ is a strongly ISP-ring if and only if A is an ISP-domain and E a simple A-module.

Strongly ISP-rings

Remark

In general, $A \propto E$ need not be a strongly ISP-ring. Indeed, let A be an ISP-domain, $K = qf(A)$ and E a K-vector space such that $dim_K(E) > 1$. By Proposition, $A \propto E$ is not a strongly ISP-ring.

The following result studies the strongly ISP-ring property for amalgamated duplication ring $A \bowtie I$.

Theorem

Let A be a ring and I an ideal of A.

1. If $A \bowtie I$ is a strongly ISP-ring, then so is A.
2. If I is a finitely generated idempotent ideal of A, then $A \bowtie I$ is a strongly ISP-ring if and only if so is A.

We conclude by giving an example of a ring A that is a strongly ISP-ring while $A \bowtie I$ is not.

Example

Let F be a field, $A = F \propto F$ and $I = 0 \propto F$ an ideal of A. Then A is a strongly ISP-ring, by the previous Proposition. Notice that $A \bowtie I \simeq A \propto I$. Hence, by the Example given before, $A \bowtie I$ is not a strongly ISP-ring since A is not a reduced ring.

Perspectives

To end this talk, we present some perspectives:

1. Extend the study of ISP-rings to the general framework of amalgamated rings along an ideal.
2. Investigate both ISP-rings and strongly ISP-rings in the context of graded ring theory.

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Thank You So Much!