Isonoetherian modules

Jan Žemlička

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Outline

Terminolgy and examples

In the sequel
- R denotes an associative ring with unit,
- M a right R-module,
- a group means an abelain group (i.e. $\mathbb{Z}$-module)

A right module M over R is called *isonoetherian* if for every increasing chain $M_0 \leq M_1 \leq \dots$ of submodules there exists n such that $M_m \cong M_{m+1}$ for each $m \geq n$.
A ring R is *right isonoetherian* if R_R is isonoetherian.

Example

(1) Every noetherian module is isonoetherian

(2) If D is a DVR with the fraction field Q then $\begin{pmatrix} D & Q \\ 0 & D \end{pmatrix}$ is a right isonoetherian ring which is not right noetharian.

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Tools

Lemma (Facchini, Nazemian)

A module M is isonoetherian if and only if, for every non-empty set $\mathcal{F}$ of submodules of M, there exists $N \in \mathcal{F}$ such that $L \cong N$, for every $L \geq N$.

Lemma

Let M be an isonoetherian module. Then there exist a finitely generated submodule F and a chain of submodules $(M_i \mid i < \omega)$ such that each finitely generated submodule containing F is isomorphic to F and

(1) *each finitely generated submodule of M/F is noetherian,*

(2) *$M_0 = F$, $M_i \subseteq M_{i+1}$ for each i and $M = \bigcup_i M_i$,*

(3) *M_{i+1}/M_i is essential in M/M_i and it is a direct sum of noetherian cyclic submodules.*

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Closure properties

Proposition

Let M be an isonoetherian module over a ring R.

(1) (Facchini, Nazemian) *M has finite Goldie dimension,*

(2) *if R semilocal, then $\mathrm{gen}(M) \leq \omega$,*

(3) *if R is commutative and κ is an infinite cardinal grater then the cardinality of the set of all maximal ideals, then $\mathrm{gen}(M) \leq \kappa$.*

Proposition (Facchini, Nazemian)

(1) *Being isonoetherian is a Morita invariant property of modules,*

(2) *the product $\prod_{i \in I} R_i$ of rings is right isonoetherian if and only if I is finite and every R_i is right isonoetherian.*

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Chain rings

Lemma (Facchini, Nazemian)

A right chain domain is right isonoetherian if and only if it satisfies the ACC on infinitely generated right ideals.

Theorem (Facchini, Nazemian)

If a commutative valuation domain R is isonoetherian, then R has at most three prime ideals and P_P is a principal ideal of the localization R_P for every prime ideal P.

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Every discrete valuation domain of Krull dimension 2 i is isonoetherian.

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Perfect rings

Lemma

If M is an isonoetherian module over a semilocal ring, then for every submodule N there exist a finitely generated submodule $F \subseteq N$ such that N is countably generated and $N/F = (N/F)J$.

Theorem

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Abelian groups

Lemma

Let M be an isonoetherian group with the torsion part T. Then

(1) *T is finite and there exist a torsion-free submodule N of a finite rank such that $M = F \oplus T$,*

(2) *if F is a free subgroup of M such that $\mathrm{rank}(F) = \mathrm{rank}(M)$, then M/F contains unbounded p-subgroups for only finitely many prime numbers p.*

Proposition

Let F be a free group of finite rank, and M be any group. Then M is isonoetherian if and only if $M \oplus F$ is isonoetherian.

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Necessary conditions

Let M be a torsion-free group of a finite rank and $F \leq M$ a subgroup of the same rank. Denote by $d_p(M)$ the rank of p-component of the divisible part of M/F.

Proposition

Let M be a torsion-free isonoetherian group with finitely generated subgroup F such that $M/F \cong \mathbb{Z}^2_{p^\infty}$. Then $\exists$ indecomposable subgroups $A_1, A_2 \leq M$ such that $d_p(A_1) = d_p(A_2) = 1$, $d_p(A_1 + A_2) = 2$. Let A_1 and A_2 be such and $A = A_1 + A_2$. Then $\exists\, \alpha \in \mathrm{Aut}(A)$, and a sequence of finitely generated subgroups $F_0 \leq F_1 \leq \dots$ such that

1. *$\tilde{\alpha} = p \cdot \mu$ for algebraic $\mu \in \hat{\mathbf{Z}}_p^*$ where the endomorphisms $\tilde{\alpha} \in \mathrm{End}(A/F_0)$ is induced by α,*
2. *$\alpha(C) = C$ for each $C \leq A$ which is indecomposable and pure and $d_p(C) = 1$,*
3. *$A_1 \cap A_2 \leq F_0$, $\alpha(F_i) = F_{i-1}$ for all $i > 0$, and $\alpha(A_1) = A_1$, $\alpha(A_2) = A_2$, $\alpha(A_1 \cap A_2) = A_1 \cap A_2$.*

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Decomposable case

Proposition

Let p be prime and A an indecomposable torsion-free group such that $\mathrm{rank}(A) > 1$ and $A/G \cong \mathbb{Z}_{p^\infty}$ for a finitely generated subgroup G . Then $A \oplus \mathbb{Z}[\frac{1}{p}]$ is not isonoetherian.

Theorem (Keef)

Suppose G is a completely decomposable torsion-free group of finite rank. Then G is isonoetherian exactly in the following three cases:

(1) $G \cong \mathbb{Z}[1/(p_1 \cdots p_k)] \oplus A$, *where the ps are distinct primes and A is free.*

(2) $G \cong \mathbb{Z}[1/p]^2 \oplus A$, *where p is a prime and A is free.*

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